

**THE BULLETIN OF THE**



**USER GROUP**

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**Especially for our German DUG-Members!**  
**Besonders für unsere deutschen DUG-Mitglieder!**

Für das Frühjahr 1993 planen wir ein Treffen der deutschen Mitglieder der DERIVE USER GROUP.

Ich schlage als Termin den Samstag, den 24. April 1993 vor. Als Ort der Zusammenkunft hat Hr. Appel aus Kronach Schweinfurt angeboten. Hr. Appel hat auch bereits freundlicherweise Vorsorge für Nächtigungsmöglichkeit, ein Tagungslokal und für die PC-Ausstattung getroffen. Herzlichen Dank dafür, lieber Herbert.

Ich könnte mir vorstellen, dass wir uns am Freitag, den 23. April abends in Schweinfurt treffen und dass wir dann am Samstag DERIVE-bezogene Themen behandeln. Bitte teilen Sie mir mit der beigelegten Information für den Herausgeber des DNL mit, ob Sie an diesem Treffen teilnehmen wollen (mit oder ohne Nächtigung in Schweinfurt). Teilen Sie mir bitte weiters mit, ob Sie bereit wären, einen Kurzvortrag (kurze Themenbeschreibung und Dauer) im Rahmen des 1. Deutschen DUG-Treffens zu halten. Als Meldeschluss bitte ich den 31. Jänner 1993 zu beachten. Sie werden anschließend umgehend über das Zustandekommen und über das Programm verständigt. Das Treffen in Birmingham im Frühjahr war eine wirklich gelungene Veranstaltung. Ich lade Sie herzlich ein, sich am diesem Treffen in Schweinfurt zu beteiligen. Natürlich sind auch unsere Freunde, die nicht in Deutschland wohnhaft sind herzlich willkommen, wenn sie die Anreise nicht scheuen. Also bitte. denken Sie an den 31. Jänner 1993.

D E R I V E - B O O K - S H E L F

**Hoogewijis A**, DERIVE, Een Steun bij het Wiskunde-Onderwijs? (DERIVE, A Support in Teaching Mathematics?) in Wiskunde en Onderwijs 18 (1992)

**AMMU** (Arbeitsgruppe Moderner Mathematik Unterricht - Working Group Modern Mathematics Teaching). A group of Austrian teachers deals with Information Technology and Teaching Mathematics, Computer Algebra Systems (especially DERIVE), Working with Graphing Calculators, Didactics, etc.  
1<sup>st</sup> issue in Oct. 1992, 82 pages.  
Contact: Dr. Peter Schüller, A-2351 Wr. Neudorf, Reisenbauerring 6/1/35, AUSTRIA.

New information for the revised version:

**AMMU website:** [www.ammu.at](http://www.ammu.at)

Download all *DNL-DERIVE*- and TI-files from

<http://www.austromath.ac.at/dug/>

<http://www.bk-teachware.com/main.asp?session=375059>

Lieber Derive Anwender!

Das zweite Bestandsjahr der DUG ist nun abgelaufen und wenn ich vor einem Jahr stolz von 205 Mitgliedern berichten konnte, so sind wir nun bereits eine Gemeinschaft von über 520 Derivianern auf der ganzen Welt geworden. Fast täglich erreichen mich Anmeldungen und Anfragen. Die aktuelle "Verteilungsfunktion" finden Sie im Kasten. Ein Blick auf die Beiträge in diesem DNL, sowie eine kurze Vorschau auf die nächsten Beiträge zeigt die erfreuliche Internationalität unserer Gruppe: Mechanics (Finnland), Extended Fourier Series (Brasilien), Physics (Kanada), Analytische Geometrie im  $R^3$  (Österreich), Iterationsverfahren (Deutschland), Ideen für die Schule (Hawaii). Aus der CSFR und aus Ungarn ist Mitarbeit angekündigt worden.

Dear Derive User,

The DUG has now been existing for two years and when I could proudly report in the Editor's Letter about 205 members one year ago so we have grown up to now to a community of more than 520 Derivians from all over the world. Nearly every day I receive applications or/and inquiries. You will find the actual "distribution function" in the box. One glance to the contributions of this DNL and a short preview to next articles show the pleasing internationality of our group: Mechanics (Finland), Extended Fourier Series (Brasil), Physics (Canada), Iterations (Germany), Analytical Geometry (Austria), Ideas for the Classroom (Hawaii). Users from the CSFR and from Hungary have announced their cooperation.

Argentina 1	Germany 160	Portugal 4
Australia 5	Hongkong 1	Slovenija 1
Austria 76	Ireland 2	Spain 7
Belgium 3	Italy 17	Sweden 23
Brasil 2	Luxemburg 2	Switzerland 23
Canada 3	Mexico 1	United Kingdom 57
Danmark 2	Netherlands 18	USA 81
Finland 4	Norway 6	
France 25	Poland 1	

Bitte beachten Sie die beiliegenden Informationen betreffend die Konferenz in Krems 1993 und die Bezugsmöglichkeit des Tagungsbandes der Spring School 1992 in Krems.

Sie finden auch ein Papier für eine Rückmeldung an mich. Bitte nutzen Sie diese Gelegenheit, damit wir Ihnen mit dem DNL und der DUG eine noch bessere Serviceleistung bieten können. Ich ersuche Sie natürlich wieder, Beiträge für den DNL zu liefern oder anzuregen.

Bitte begleichen Sie rechtzeitig den Mitgliedsbeitrag für 1993. Der Verwaltungsaufwand ist bereits recht groß und ich danke für Ihr Verständnis im Namen meiner Frau, die mir die administrative Arbeit abnimmt.

Mit den besten Grüßen und Wünschen zu den Festtagen und zum Jahreswechsel



Please pay attention to the information concerning the 2<sup>nd</sup> Krems Conference on Mathematics Didactics and the possibility to order the Proceedings of the Spring-school 1992 in Krems.

You will find another paper which should be an information for me. Please take the opportunity of a feedback so that we are enabled to offer more and better services with DNL and DUG. I request for deliverance of contributions and for suggestions once more.

Please settle your dues for 1993 in time. The administration is becoming enormous and I thank for your comprehension in my wife's name for she does all the administration work.

With best wishes and greetings for Christmas and for the New Year 1993



The *Derive-News-Letter* is the Bulletin of the Derive-User Group. It is published at least three times a year with a contents of 30 pages minimum. The goals of the *D-N-L* are to enable the exchange of experiences made with Derive as well as to create a group to discuss the possibilities of new methodical and didactic manners in teaching Mathematics.

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### Contributions:

Please send all contributions to the above address. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *D-N-L*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *D-N-L*. The more contributions you will send to the Editor, the more lively and richer in contents the *Derive-News-Letter* will be.

### **Preview: (Contributions for the next issues)**

Newton's Chaos;

Extended Fourier Series;

.....Physics in the Classroom

The Mechanics of Erkki Ahonen;

The Triangle of PASCAL;

(will be published March 93)

### **Impressum:**

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**Terence Etchells, Bolton, UK**

Thanks for the last edition of the DUG I have enjoyed reading it immensely, especially your article on Riemann at Random. As you know I have done some work in this area and Paul Drijvers pointed out that my functions (and presumably yours) only worked for monotonic functions. I decided on my return to the U.K. that I would try to write a function that would give true Riemann rectangles. Your article has motivated me to write an article on the functions that I wrote, enclosed with this letter. I would be most grateful if you would consider publishing it in a future edition of the DNL. I have gone some pain to try to explain how the functions work. I sent a copy of the functions to Paul and he wrote back commenting that he did not understand how the routine works although he did find it beautiful!

**DNL:** Your article is published in this issue. So are my comments and your answer. I think its a fine example for top - down with DERIVE!

**Helmut Dieck, Essen, Germany**

Inzwischen kann ich einigermaßen mit dem Programm umgehen, aber es sind doch noch ein paar Fragen offen geblieben.

1. Mir fiel auf, dass das Programm die Gleichung  $\text{SQRT}(x) = a$  nicht nach  $x$  auflösen kann, es sei denn, man setzt für  $a$  eine konkrete Zahl (z.B.:  $a = 3 + 4i$ ) ein. Warum?
2. Zu einer Gleichung 4. Grades fand es "Lösungen", die keine sind: siehe die beiden Anlagen A1 und A2. Habe ich da einen Fehler gemacht? Wenn es ein Fehler des Programms ist, welche Fehler hat es noch?

Mr Dieck wants to know (1) why the equation  $\text{SQRT}(x) = a$  cannot be solved for  $x$  and he (2) created an equation of degree 4. But solving this equation solutions appeared, which are no solutions. See the following example:

$$(z^4 - 20z^3 - 66z^2 + 1420z - 2315 - 4i) (z^4 - 8z^3 - 85z^2 + 184z + 275) = 0$$

and after a lot of while solving in Exact Mode (about 1000 seconds calculation time) he finally got 20 lines of solutions and after approximating *DERIVE* gave:

$$z = 3 + 4i; z = 7 + 8i; z = 3 + 4i; z = 7 + 8i$$

**DNL:** For your first question see the following hint please:

- 1:  $\sqrt{x} = a$
- 2: "Declare Variable x Complex"
- 3:  $x = a^2$

The recent versions of *DERIVE* give more sophisticated answers:

<b>P 4</b>	<b>DERIVE – USER – FORUM</b>	<b>D-N-L#8</b>
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```

#1: SOLVE( $\sqrt{x} = a$ ,  $x$ ) = ( $x = \text{IF}(a > 0, a^2)$ )
#2:  $a \in \text{Complex}$ 
#3: SOLVE( $\sqrt{x} = a$ ,  $x$ ) =  $\left( x = \text{IF}\left(-\frac{\pi}{2} \leq -\text{PHASE}(a) < \frac{\pi}{2}, a^2\right) \right)$ 
#4:  $a \in \text{Real } (0, \infty)$ 
#5: SOLVE( $\sqrt{x} = a$ ,  $x$ ) = ( $x = a^2$ )

```

Derive 6.10

DERIVE 6.10 has also problems in factorizing the quartic in Exact Mode. Using a numerical algorithm DERIVE is able to solve the equation to our satisfaction:

```

#3: SOLVE( $z^4 - 20 \cdot z^3 - 66 \cdot z^2 + 1420 \cdot z - 2315 - i \cdot (24 \cdot z^3 - 340 \cdot z^2 + 736 \cdot z + 1100)$ ,  $z$ )
#4: false
#5: SOLVE( $z^4 - 20 \cdot z^3 - 66 \cdot z^2 + 1420 \cdot z - 2315 - i \cdot (24 \cdot z^3 - 340 \cdot z^2 + 736 \cdot z + 1100)$ ,  $z$ )
#6:  $z = 2.35776 + 4.99075 \cdot i \vee z = 6.00924 + 8.64223 \cdot i \vee z = 3.99075 + 3.35776 \cdot i \vee z = 7.64223 + 7.00924 \cdot i$ 
#7: NSOLVE( $z^4 - 20 \cdot z^3 - 66 \cdot z^2 + 1420 \cdot z - 2315 - i \cdot (24 \cdot z^3 - 340 \cdot z^2 + 736 \cdot z + 1100)$ ,  $z$ )
#8:  $z = 8 + 9 \cdot i \vee z = 6 + 7 \cdot i \vee z = 4 + 5 \cdot i \vee z = 2 + 3 \cdot i$ 

```

### Helmut Scheuermann, Hofheim/Taunus, Germany

Helmut Scheuermann sent an equation which he wanted to solve with *DERIVE* 2.06. An equation which gave no problems using version 1.55 was not to solve with the later version.

1: "Version 1.55"	1: "Version 2.05"
2: $\frac{x + \text{LN}(x)}{x^a} = \frac{x + \text{LN}(x)}{x^b}$	2: $\frac{x + \text{LN}(x)}{x^a} = \frac{x + \text{LN}(x)}{x^b}$
3: "soLve"	3: "soLve"
4: $x = 1$	4: $\text{LN}(x) = \frac{1}{0}$
5: $\text{LN}(x) + x = 0$	
6: "soLve in approXimate Mode:"	5: $\frac{b}{x (\text{LN}(x) + x)} - \frac{a}{x (\text{LN}(x) + x)} = 0$
7: $x = 0.567143$	

**DNL:** I wrote the story of this equation from V. 1.06 up to V. 2.52. Version 2.10 and later is able to solve the equation. It seems that some versions have forgotten a part of their earlier knowledge.

This is the answer of *DERIVE* 6.10 in exact and approximate mode in 2005:

$$x \cdot e^x = 1 \vee x = 1 \vee a - b = 0$$

$$x = \text{IF}(a < 0, 0) \vee x = \text{IF}(b < 0, 0) \vee x = \text{IF}(a > 0, \infty) \vee x = \text{IF}(b > 0, \infty) \\ \vee x = 0.5671432986 \vee x = 1 \vee a - b = 0$$

### Herbert Appel, Kronach, Germany

Answer to Mr.Neuraths's letter in DNL#7:

Thank you for the hint in DNL#7. When I said "pupils want to test the teachers competence in using the program" I meant that the students try to disturb the lesson by changing some menu-options (O-P,M-T,W-D etc.). This is especially then annoying when the correct choice is important for the solution of a problem on this level. Another example: When I want to teach to solve an equation step by step, it is important that the children are not able to use the command SOLVE. Other menu-points are not used in secondary schools at all. For example: Calculus, Declare-3DPlot, Complex Numbers and so on. Of course it is a welcome effect that the pupils are free in the selection of the way to solve a mathematical problem, but the experience (with other programs) shows that the pupils are less confused if only the menu-points needed are shown on the display. Yours faithfully H. Appel

### Günter Scheu, Pforzheim, Germany

The first name of Ch. Goldbach was not Charles but Christian.

See H.Meschkowski, Mathematiker Lexikon:

(I found *Charles* in a textbook dealing with number theory. Thank you Günter for giving Goldbach his correct Christian name.

#### **GOLDBACH, CHRISTIAN (18. 3. 1690 – 12. 1. 1764)**

*Sohn eines Königsberger Pastors. Studierte Jura und hatte seine Freude an der Mathematik. Auf Reisen durch Europa traf er mit führenden Mathematikern seiner Zeit zusammen (LEIBNIZ; NIKOLAUS BERNOULLI u.a.) und blieb mit vielen von ihnen im Briefwechsel, als er Sekretär der Petersburger Akademie wurde.*

In den Protokollen der Akademie finden sich häufig Hinweise auf mathematische Mitteilungen GOLDBACHS, z.B. über Fragen der Reihenlehre, über die Riccatische Differentialgleichung.

Auch der jetzt herausgegebene Briefwechsel mit EULER enthält Mitteilungen, die von dem mathematischen Verständnis GOLDBACHS und seinem Ideenreichtum zeugen. Besonders wichtig ist die bis heute noch nicht bewiesene „Goldbachsche Vermutung“: Jede gerade Zahl ist als Summe zweier Primzahlen darstellbar.

Message 1864 was entered on 5/14/92 at 12:16 PM. (Read 31 times)

**From JASON to PUBLIC about EXPONENTIAL EXPAND**

I am looking for a way to expand exponentials to exponentials instead of sines and cosines. Is there a way? I tried changing the branches but doesn't help. I need this ability to calculate Fourier Transforms.

Thanks Jason B. B.

P. S. I'm still looking for the difference between Derive VER 2.06 and 2.10

<<< Reply - see message #1865 >>>

Message 1865 was entered on 5/14/92 at 3:58 PM. (Read 30 times)

**From JERRY GLYNN to JASON about #1864 / EXPONENTIAL EXPAND**

Try `MANAGE-EXPONENTIAL-EXPAND-ENTER` and then `SIMPLIFY`.  $e^{(x+y)}$  becomes  $e^x * e^y$ .

Message 1888 was entered on 5/19/92 at 7:00 AM. (Read 29 times)

**From JASON to JERRY GLYNN about #1866 / EXPONENTIAL EXPAND**

Jerry, thank you for the reply. I think I found my problem in that when I use `exp(#i)`, it automatically simplifies to sines and cosines. If I use the `j` operator instead of `#i` then I can simplify down with exponentials. I use the `manage substitute` command to replace `j` with `#i` before final simplification. This takes care of my problem. Thanks JBB

Message 2071 was entered on 6/30/92 at 12:12 PM. (Read 21 times)

**From GREG SMITH to PUBLIC about DERIVE/DR-DOS**

Is anyone running Derive (successfully or unsuccessfully) under DR-DOS 6.0? I know there were some compatibility problems with earlier versions, and wondered if these had been fixed in the latest DR release. A user on one of my other BBSs needs information about this.

<<< Reply - see message #2082 >>>

Message 2082 was entered on 7/4/92 at 12:06 PM. (Read 16 times)

**From JIM POTTER to GREG SMITH about #2071 / DERIVE/DR-DOS**

I have had no problems using Derive with DR-DOS 6.0. I have the latest upgrade that makes it compatible with Windows (Although that is probably not relevant to DERIVE.). I am using SuperStor 2.0 and QEMM instead of the disk compression and memory managers that come with DR-DOS. Good Luck.

Message 2147 was entered on 7/21/92 at 10:48 PM. (Read 17 times)

**From SOFT WAREHOUSE to PUBLIC about EQUATION SOLVING BY BISECTION**

In approximate mode, DERIVE uses efficient numerical techniques to compute an approximate real solution of an equation containing only one variable (see Section 4.14 in the DERIVE User Manual).

For pedagogical reasons, some users have requested a function that would show successively more accurate approximate solutions to an equation. If  $u(a)$  and  $u(b)$  have different signs, the function **BISECT**( $u, x, a, b, n$ ) defined as follows approximates to  $n$  successively more accurate solutions of the equation  $u(x)=0$ . Note that  $u(x)$  should NOT be an equation itself.



BISECT\_AUX(u, x, a, b, c) := IF(LIM(u, x, a)·LIM(u, x, c) < 0, [a, c], [c, b])

BISECT(u, x, a, b, n) := ITERATES(BISECT\_AUX(u, x, ELEMENT(v,1), ELEMENT(v,2), (ELEMENT(v,1) + ELEMENT(v,2))/2), v, [a, b], n)

BISECT(x^3 - 5·x^2 + 2, x, 0, 1, 5)

0	1
0.5	1
0.5	0.75
0.625	0.75
0.625	0.6875
0.65625	0.6875

Message 2168 was entered on 7/27/92 at 10:52 PM.

**From SOFT WAREHOUSE to PUBLIC about ALGEBRAIC SUBSTITUTION**

Some users have had difficulty doing the following exercise presented page 56 of the *DERIVE* User Manual: Use the Manage Substitute command to algebraically substitute  $e$  for  $m c^2$  in the expression

$$\frac{g m c^2}{\text{SQRT}(m^2 c^4 + 1)}$$

The key to successfully performing this exercise is the need to ALGEBRAICALLY substitute for  $m c^2$  since syntactic substitution would not replace the  $m^2 c^4$  term in the expression. The easiest way to do make this algebraic substitution is to substitute  $e/c^2$  for  $m$  in the original expression. This substitution is found by solving the equation  $e = m c^2$  for  $m$ .

To actually make the substitution, highlight the ENTIRE expression and then issue the Manage Substitute command. When the command asks for a replacement for  $m$ , enter the value  $e/c^2$ . Simplifying the resulting expression should give you the expression #2 as shown on page 56.

$$\text{SOLVE}(e = m \cdot c^2, m) = \left( m = \frac{e}{c^2} \right)$$

$$\text{SUBST} \left( \frac{g \cdot m \cdot c^2}{\sqrt{(m^2 \cdot c^4 + 1)}}, m, \frac{e}{c^2} \right) = \frac{e \cdot g}{\sqrt{(e^2 + 1)}}$$

Message 2238 was entered on 8/19/92 at 1:47 AM.

**From SOFT WAREHOUSE to PUBLIC about LEAST-SQUARES RESIDUALS**

File **RESIDUAL.MTH** includes a function that computes residuals for a 2D data fit computed by the FIT function. The file also includes an example of using RESIDUAL.

Advice about least-squares fitting:

It is generally unwise to have more terms than the least of

- a) the square root of the number of data rows,
- b) the square root of Option Precision Digits.
- c) the estimated number of significant digits in the data.

It is wise to determine these terms incrementally as follows:

1. Plot the data.
2. Guess a good form for the dominant term and do a corresponding fit.
3. Plot the residuals.
4. Quit if they are as small as your estimated random experimental errors.
5. Reject the most recent term if the residuals are not significantly smaller than the original ordinates or the previous residuals.
6. Guess a good form for the dominant term of the residuals and do a revised fit of the original data including this additional term.
7. Go to step 3.

*You will find this complete file on the disk of the year '92.*

Message 2241 was entered on 8/20/92 at 12:47 AM. (Read 11 times)

**From SOFT WAREHOUSE to PUBLIC about CONVOLUTION INTEGRALS**

At a recent trade show we were asked if Derive could do convolution integrals. We looked up the definition in an electrical engineering text at the adjacent booth, and it can be implemented as follows:

To convolve an expression  $u(t)$  with an expression  $v(t)$  over the interval  $a \leq t \leq b$ , define

$$\text{CONVOLUTION } (u, v, t, a, b) := \text{INT } (u, \text{LIM } (v, t, t-x), x, a, b)$$

Usually,  $v$  is  $\exp(t)$ , in which case we can use the more straightforward definition

$$\text{CONVOLUTION } (u, t, a, b) := \text{INT } (u, \exp(t-x), x, a, b)$$

Message 2267 was entered on 8/26/92 at 8:11 AM. (Read 15 times)

**From JERRY GLYNN to PUBLIC about TEACHING GRAPHING**

In Derive we can plot points in 2 dimensions by plotting  $[a,b]$ . With some of my students who are beginning graphing I suggest that they plot a number of points (somewhat at random). If I show them  $[2,3]$  as an example they often plot other points with a pair of positive numbers. I may then ask if they can plot a point "over here" pointing to the 2nd quadrant or 3rd or 4th. If they experiment and make it fine...if not I may a suggestion (try a negative number). I also show them how to clean the screen and how to retrieve earlier points.

Later I ask if they can plot some points in a row (I may also show them how to "back up theit viewing helicopter" by using F10). With a clean screen I suggest they plot  $x$  and  $2x$  and  $3x$  and leave them to experiment. Can you get a graph to slant this way or that way...

With a clean screen plot a point (maybe it's a planet) and can you plot a path through this planet (trial and error all the way)...how about 2 planets...3...have you tried to plot  $xx$  or  $xx-3$ ? And on and on.

Message 2273 was entered on 8/27/92 at 9:18 PM. (Read 12 times)

**From KAREN to JERRY GLYNN about #2267 / TEACHING ALGEBRA I WITH DERIVE**

Jerry - I'm back on line after being gone for too long. School started yesterday and I used DERIVE today with my ALGEBRA I students. I have several discovery lessons written and plan to let the students discover patterns rather than memorize rules. I taught at EXETER computer Conference this summer using DERIVE and ALGEBRA MENTOR. It went well and the participants were excited about using a symbolic manipulator with young high school students.

Message 2282 was entered on 9/1/92 at 1:56 AM.

**From SOFT WAREHOUSE to PUBLIC about CYLINDRICAL COORDINATES**

File **CYLINKIN.MTH** defines functions that compute the radial, azimuthal and axial speed and acceleration of a point  $p(t) = [r(t), \theta(t), z(t)]$  moving through radian cylindrical coordinates as a function of time  $t$ . This file can be regarded as additions to the distributed utility file **DIF\_APPS.MTH**.

*You will find this complete file on the disk of the year '92.*

Message 2289 was entered on 9/3/92 at 2:24 AM. (Read 26 times)

**From SOFT WAREHOUSE to PUBLIC about CANTILEVER BEAMS**

File **CANTILEV.MTH** defines functions that compute the shear, bending moment, slope and deflection of a cantilever beam having variable load, Young's modulus and moment of inertia. The file also contains examples of their use.

*You will find this complete file on the disk of the year '92.*

Message 2293 was entered on 9/4/92 at 12:01 AM. (Read 25 times)

**From SOFT WAREHOUSE to PUBLIC about KINEMATICS IN SPHERICAL COORDINATES**

File **SPHERKIN.MTH** defines functions that compute the radial, longitudinal and colatitudinal speed and acceleration of a point  $r(t) = [\text{radius}, -\text{longitude}, \text{latitude}]$  moving through radian spherical coordinates as a function of time  $t$ . This file can be regarded as additions to the distributed utility file **DIF\_APPS.MTH**.

*You will find this complete file on the disk of the year '92.*

Message 2295 was entered on 9/4/92 at 8:46 AM. (Read 30 times)

**From BENOITF to JERRY GLYNN about OPTIONS/EXECUTE COMMAND**

I have Derive v2.06 and when I try to execute a DOS command it tells me "Cannot find COMMAND.COM". My command.com is defined in COMSPEC. I tried to put command.com in my default directory (where is DERIVE.EXE) and I have the same problem. What is the problem?

<<< Reply - see message #2296 >>>

Message 2296 was entered on 9/4/92 at 9:51 AM. (Read 27 times)

**From GREG SMITH to BENOITF about #2295 / OPTIONS/EXECUTE COMMAND**

If you haven't already, try putting a copy of COMMAND.COM in the root directory. Programs SHOULD try to read the COMSPEC environment variable rather than assuming the name and location of the command processor, but many do not. Looking for COMMAND.COM in \ (root) is one of the common older strategies.

p10	From the <i>DERIVE</i> Bulletin Board Service	D-N-L#8
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Message 2311 was entered on 9/9/92 at 1:16 AM. (Read 23 times)

**From SOFT WAREHOUSE to PUBLIC about EULERIAN ANGLE KINEMATICS**

File **EULERANG.MTH** defines angular velocity in terms of the Eulerian angles: precession, nutation and spin. The file also includes an example.

*You will find this complete file on the disk of the year '92.*

Message 2333 was entered on 9/19/92 at 10:22 PM. (Read 16 times)

**From RANDY to PUBLIC about DERIVE & INEQUALITIES**

Does anyone know how (or if you can) plot linear inequalities or systems of linear inequalities on Derive. I'm teaching a business linear algebra course and I think it would be helpful.

<<< Reply - see message #2335 >>>

Message 2335 was entered on 9/20/92 at 7:45 AM. (Read 16 times)

**From JERRY GLYNN to RANDY about #2333 / DERIVE & INEQUALITIES**

I agree and have requested same for quite a while. Dave Stoutemyer has expressed concern that the fill algorithm he would use to indicate which side of a line was true might "leak" and fill the entire screen if the line had a hole in it (like  $(x^2-5x+6)/(x-3)$ ). Maybe we all need to ask again and I guess you have so thanks.

<<< Reply - see message #2336 >>>

Message 2336 was entered on 9/20/92 at 3:32 PM. (Read 18 times)

**From DAN to JERRY GLYNN about #2335 / DERIVE & INEQUALITIES**

I agree also! I know that I have expressed this a few years ago. It would be nice and practical from a teacher's point of view to be able to solve "Absolute Value" in terms both of  $<$   $>$   $<=$  and  $>=$  both linear and polynomial inequalities as well as rational inequalities. I know, memory is a problem, but surely, linear and quadratic inequalities with or without absolute values would help the Intermediate Algebra student check or verify solutions after a study in "Cut-Points" is presented. I would really like to see this implemented in "Derive".

I am convinced, that Derive is a great program for students and teachers of Mathematics. Don't leave home without it!

<<< Reply - see message #2347 >>>

Message 2347 was entered on 9/24/92 at 8:37 PM. (Read 12 times)

**From KAREN to DAN about #2336 / DERIVE & INEQUALITIES**

I teach Algebra in a high school using DERIVE. Instead of fill which might cause leaks why not have DERIVE shade inequalities the way we teach our student - Draw diagonal lines. This eliminates the leaking problem and still effectively shades the solution region so a beginning student can visualize the "answer".

## Rational Collocation

A. Marlewski, Pozna'n

1. We deal with the rational function  $F$  of the form  $F(x) = N(x)/D(x)$ , where the nominator  $N(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$  and the denominator  $D(x) = a_{n+1} + a_{n+2}x + \dots + a_{p-1}x^{p-n-2} + x^{p-n-1}$  are the polynomials, the degrees of which (respectively not greater than  $n$  and equal to  $p - n - 1$ ) are given in advance. We want to determinate their  $p$  coefficients  $a_0, a_1, \dots, a_{p-1}$  such that at every given point  $(x_j, y_j)$  there holds the equality

$$(1) \quad F(x_j) = y_j.$$

We suppose  $x_j \neq x_k$  for  $j \neq k$  and we call the system of these equalities as the collocation condition(s), the points  $(x_j, y_j)$  as the collocation knots.

The task considered by us is known as the  $(p, n)$ -th rational collocation problem or the Padé- collocation problem, because H. Padé investigated it in details in 1892. Nota bene, the first results concerning to this question were obtained some decades before, namely in 1821 by A. Cauchy and in 1846 by C. Jacobi.

2. The inquired numbers  $a_0, a_1, \dots, a_{p-1}$  are involved in the system (1). We replace it by the system

$$(2) \quad N(x_j) = y_j \cdot D(x_j), j = 1 \dots p,$$

the equations of which are the linear. Our new system is equivalent to the system (1) if every collocating knot  $(x_j, y_j)$  is accessible. Let's see the following example to understand what does the accessibility mean.

3. For  $p = 3, n = 1$  and the knots  $(1,2), (2,2)$  and  $(3,3)$  our system (2) writes

$$\begin{aligned} a_0 + a_1 - 2a_2 &= 2 \\ a_0 + 2a_1 - 2a_2 &= 4 \\ a_0 + 3a_1 - 3a_2 &= 0 \end{aligned}$$

so it yields the solution  $a_0 = -6, a_1 = 2$  and  $a_2 = -3$ , in consequence we get

$$F(x) = \frac{-6 + 2x}{-3 + x}$$

The graph of this function does not pass through the point  $(3,3)$ , the resulted function  $F$  is equal to 2 everywhere but not for the argument  $x = 3$ , where  $F$  is not defined. The point  $(3,3)$  is not accessible (for the given knots and the function of the given form). In the discussed case system (2) is not equivalent to system (1).

4. In fact, the Padé collocation problem not always has a solution. It is in contrary to the strictly polynomial collocation problem, wellknown as the Lagrange or Newton interpolating task. The Lagrange interpolation is nothing else but the Padé collocation having the denominator equal to 1, and it always has the uniquely solution under the natural assumption on the knots (we only need  $x_j \neq x_k$  for  $j \neq k$ ).

<b>p12</b>	<b>Adam Marlewski: Rational Collocation</b>	<b>D-N-L#8</b>
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5. In Point 7 you can find the unit `COL_PADE.MTH` and you can use it during your DERIVE session. This unit offers the function `COL_PADE` having two arguments. The first argument is the 2-column matrix of the given knots  $(x_j, y_j)$ , the number of the rows is the number of these knots and, as above, we denote it by  $p$ . The second argument is the degree of the denominator, it should be the natural number not greater than  $p$ . Then DERIVE transforms the expression `COL_PADE(d, n)` into one of the following results:

- a) the text "The 2nd parameter is too big.", if  $p \leq n$ ,
- b) the text "No (unique?) solution.", if the data do not define the solution (at all or uniquely); it takes place e.g., if

$$d = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \text{ and } n = 1$$

there exists no function of the form  $\frac{a_0 + a_1 x}{a_2 + x}$  passing through the points (1,1), (2,2) and (3,3),

- c) the vector  $[N(x), D(x)]$ , if there exists the point  $x_j$  such that  $D(x_j) = 0$ ; as above,  $N$  and  $D$  stand for the nominator and denominator of the function  $F$  we search, and in Point 3 there are listed the sample data invoking the situation here described,
  - d) the expression  $N(x)/D(x)$ , otherwise; this is the rational function having the denominator of degree  $\leq n$ , the denominator of degree equal to  $p-n-1$  and the given values  $y_j$  at the given arguments  $x = x_j$  with  $j = 1 \dots p$ .
6. In Point 4 we have noticed that the Pad, collocation with  $n = p - 1$  turns into the classical polynomial interpolation. The definition of the function `COL_PADE` lets to call it in this case, too, so we can obtain the polynomial  $N(x)$  of degree less than  $p$  and fulfilling the collocation condition  $N(x_j) = y_j, j = 1 \dots p$ .

Let obtain it when the data matrix is made of the knots  $(-1, 5/4), (-1/2, 32/25), (1, 1/2), (3/2, 0), (3, 1/8)$ , which are visible in Fig.1. Approximating (up to 6 digits) the expression `COL_PADE(d, 4)` gives

$$0.0508333x^4 - 0.0308333x^3 - 0.368958x^2 - 0.344166x + 1.19312.$$

This expression differs a little from the result produced by the built-in function `FIT`, which the argument is the matrix of six rows, the last five of them are the same of the matrix  $d$  and the first one (determinating the independent variable and the form of the approximating function) is  $[x, ax^4 + bx^3 + cx^2 + dx + e]$ .

(Comment: DERIVE 6 shows no difference between both results.)

7. The unit COL\_PADE.MTH:

COL\_PADE.MTH by Adam Marlewski (Poznan, Poland), 1992.07.22  
adapted for DERIVE version 6.10 by Josef Böhm, November 2005

```

A_(d,n):=VECTOR(
  [lim
  [x,y]→[d↓j↓1,d↓j↓2]
  APPEND(VECTOR(x^k,k,0,n),VECTOR(-y*x^k,k,0,DIM(d)-n-2)),j,1,DIM(d))

B_(d,n):=VECTOR(
  [d
  j,1
  DIM(d)-n-1
  d
  j,2
  j,1,DIM(d))

S_(d,n):=A_(d,n)^-1*B_(d,n)

NOM(d,n):=Σ_{k=0}^n (S_(d,n))_{k+1} * x^k

DEN(d,n):=Σ_{k=0}^{DIM(d)-n-2} (S_(d,n))_{n+2+k} * x^k + x^{DIM(d)-n-1}

COL_PADE(d,n):=
  If n ≥ DIM(d)
  "The 2nd parameter is too big."
  If DET(A_(d,n)) = 0
  "No (unique ?) solution."
  If Π(LIM(DEN(d,n),x,d↓j↓1),j,1,DIM(d)) = 0
  [NOM(d,n),DEN(d,n)]
  NOM(d,n)/DEN(d,n)

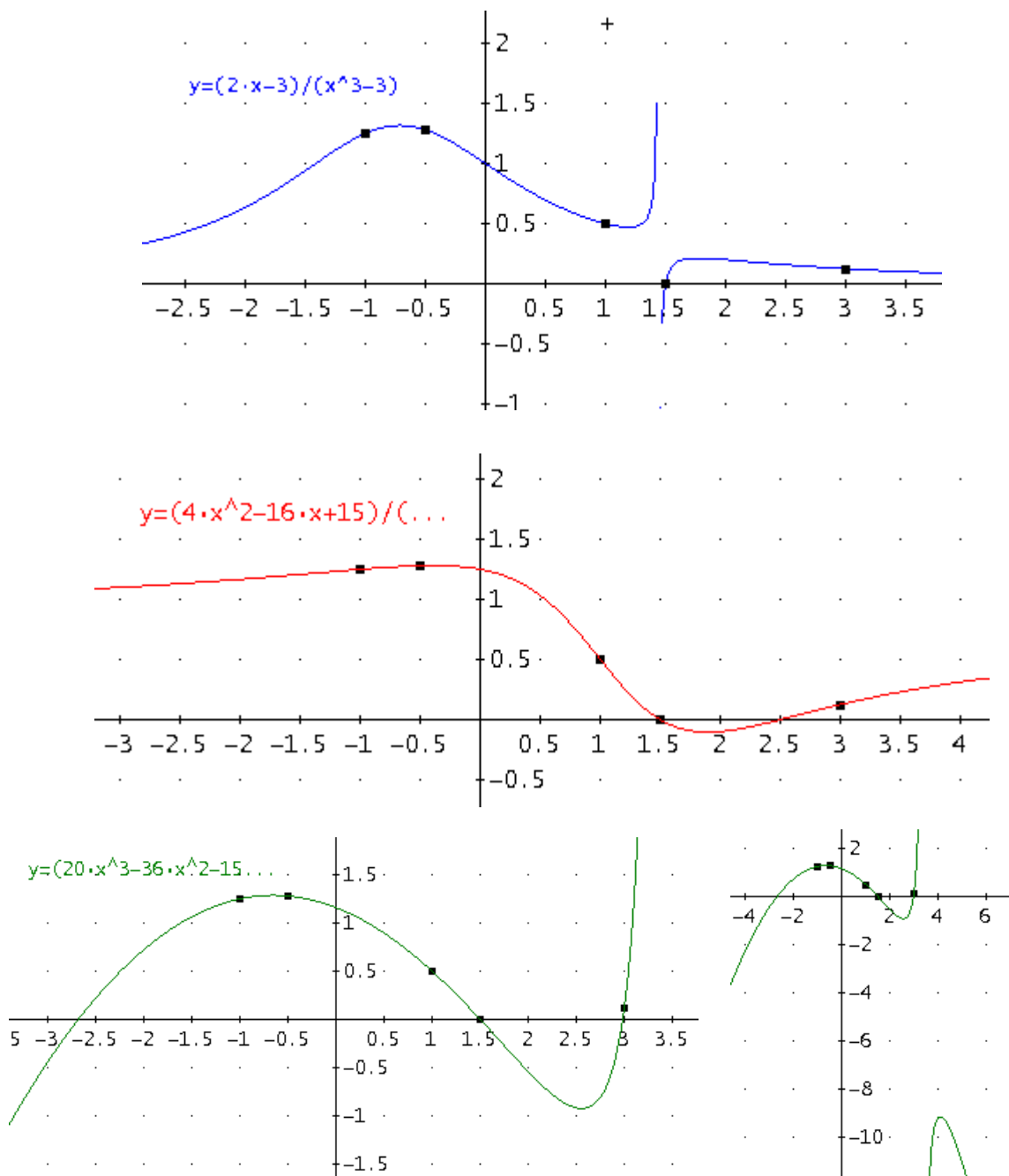
```

$$\#7: \quad d := \begin{bmatrix} -1 & \frac{5}{4} \\ -\frac{1}{2} & \frac{32}{25} \\ 1 & \frac{1}{2} \\ \frac{3}{2} & 0 \\ 3 & \frac{1}{8} \end{bmatrix}$$

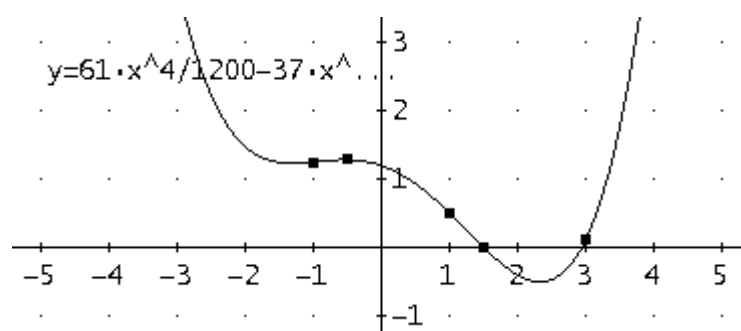
$$\begin{aligned} \text{COL\_PADE}(d, 1) &= \frac{2 \cdot x - 3}{x^3 - 3} \\ \text{COL\_PADE}(d, 2) &= \frac{4 \cdot x^2 - 16 \cdot x + 15}{5 \cdot x^2 - 11 \cdot x + 12} \\ \text{COL\_PADE}(d, 3) &= \frac{20 \cdot x^3 - 36 \cdot x^2 - 151 \cdot x + 240}{207 - 61 \cdot x} \\ \text{COL\_PADE}(d, 4) &= \frac{61 \cdot x^4}{1200} - \frac{37 \cdot x^3}{1200} - \frac{1771 \cdot x^2}{4800} - \frac{413 \cdot x}{1200} + \frac{1909}{1600} \end{aligned}$$

Some changes were necessary to make Adam's file from 1992 working with DERIVE 6. So Adam had two functions called DENOMINATOR and NUMERATOR, which are now implemented DERIVE-functions (with another functionality).

The next screen dumps show the collocation knots together with the Padé approximations.

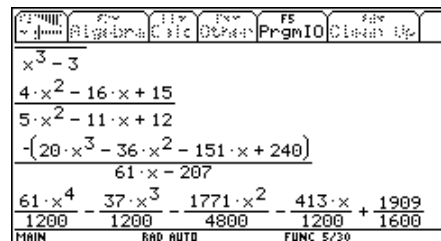
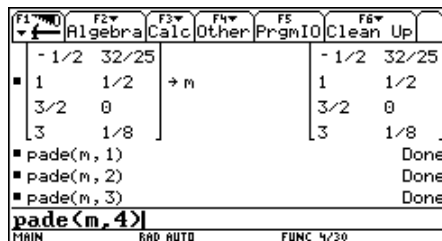


There is a pole, which can be seen by zooming out! See finally the quartic!

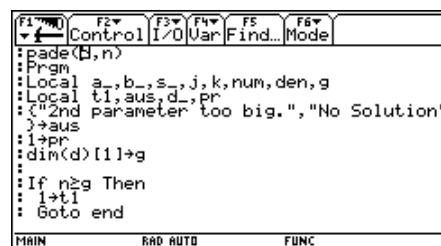
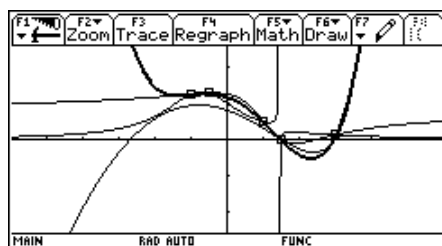




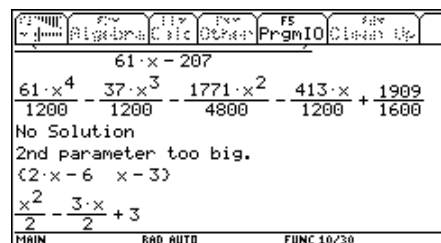
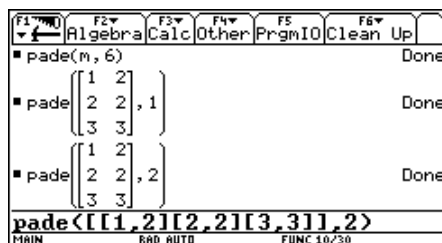
As you can see documented by the following screenshots I was successful in transferring Adam Marlewski's routine on the TI. I receive the Padé-functions in the PrgmIO-window and can plot them together with the scatter diagram given by the knots. Josef



I am editing matrix m and then calling program pade(d,n) as Adam did in Derive. The rational expressions appear on the PrgmIO Screen. The function found is saved as pf and then be stored as one of the y-expressions in the Y=-Editor. I stored the four possible functions and plotted them together with the given scatter diagram of knots.



The right screen shows the start of the Pade-programm.



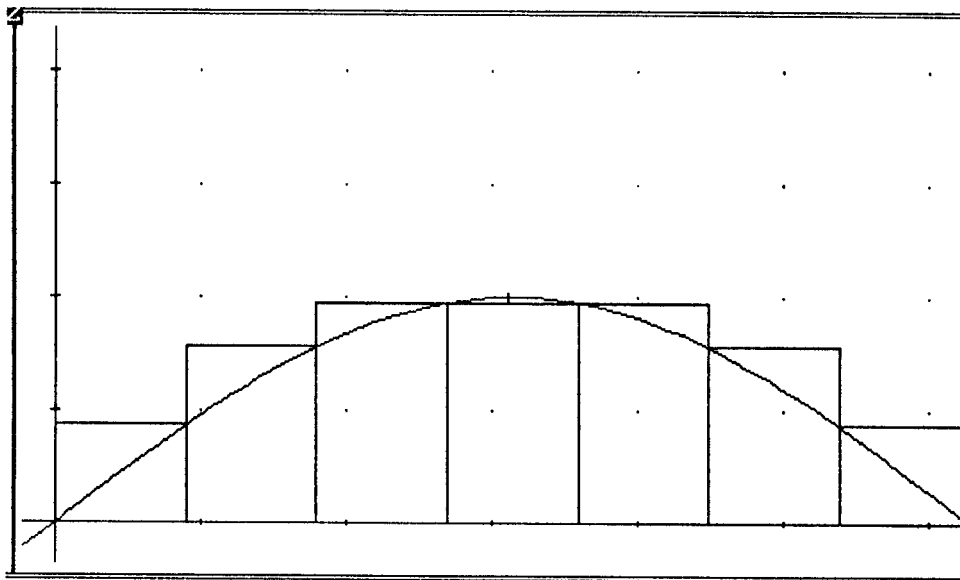
Here you can see how Adam's routine deals with questionable or „no solutions“. This is again an example for the fact that it is not very difficult to exchange ideas from one platform to the other. I am no expert on this field, I followed Adam's instructions ....

## True Riemann Rectangles

*Terence Etchells, Bolton, UK*

In the Derive Newsletter number 7, Josef Böhm described a technique for displaying and calculating Riemann rectangles of random width. I had independently written functions that would do the same thing (except for random widths) at the Spring School in Krems. Paul Drijvers, Utrecht drew my attention to the fact that the routines did not give truly upper or lower Riemann rectangles for non monotonic functions.

The screen dump below describes the problem. The function  $\sin x$  between 0 and  $\pi$  for seven rectangles:



The original DOS-DERIVE screen dump from 1992

The rectangles are generated by a VECTOR-command shown below in ASCII-format. (Now it is in DERIVE Unicode format.)

```
rect(a,b,m,n):=[ [a,b;a,b+m], [a,b+m;a+n,b+m], [a+n,b+m;a+n,b] ]
```

```
rectupper(u,a,b,n):=VECTOR(rect(a+j(b-a)/n,0,MAX(LIM(u,x,a+(b-a)*j/n),
LIM(u,x,a+(b-a)*(j+1)/n)), (b-a)/n), j,0,n-1)
```

```
rectlower(u,a,b,n):=VECTOR(rect(a+j(b-a)/n,0,MIN(LIM(u,x,a+(b-a)*j/n),
LIM(u,x,a+(b-a)*(j+1)/n)), (b-a)/n), j,0,n-1)
```

**u** is the particular function, **a** and **b** are the limits of the particular area and **n** is the number of rectangles. The procedure creates matrices that will plot the Riemann upper and lower rectangles. The interval  $[a,b]$  is cut into  $n$  equal strips, each strip is the base of a Riemann rectangle. In the case of the upper Riemann rectangles, the co-ordinates of each side of the strip are substituted into the function  $u$ , and the maximum of these two values is picked as the height of the rectangle.

If there is a turning point within a strip, then this procedure will not pick the true maximum (or minimum) over the strip's interval. Hence as in the picture above the rectangle produced is not a true upper Riemann rectangle. I left Krems with the task of writing a procedure, in Derive, that would give the true upper Riemann rectangles. I doing so I found that Derive's ability to symbolically manipulate expressions makes it a very special and unique programming language.

If there is a turning point within a strip then the sign of the derivative at each end of the strip is different. If this is the case we need to find the co-ordinate of the turning point and then substitute this value into the function  $u$  to get the height of the Riemann rectangle.

The first thing to do is to determine if there is a change of sign of derivative over a strip. If the interval is  $[a,b]$  and the interval is split into  $n$  strips then the  $x$  co-ordinate of each end of the  $j^{\text{th}}$  strip is  $a + \frac{(b-a)j}{n}$  and  $a + \frac{(b-a)(j+1)}{n}$ , we need to substitute these values into the derivative of the function.

This is done via the limit function, i.e.  $\lim_{x \rightarrow a + \frac{(b-a)j}{n}} \frac{d}{dx} u$ . By employing the IF, AND and OR functions we can determine whether there has been a sign change, i.e.

$$IF \left[ \lim_{x \rightarrow a + \frac{(b-a)j}{n}} \frac{d}{dx} u < 0 \text{ AND } \lim_{x \rightarrow a + \frac{(b-a)(j+1)}{n}} \frac{d}{dx} u > 0 \text{ OR } \lim_{x \rightarrow a + \frac{(b-a)j}{n}} \frac{d}{dx} u > 0 \text{ AND } \lim_{x \rightarrow a + \frac{(b-a)(j+1)}{n}} \frac{d}{dx} u < 0, \text{ then, else} \right]$$

If none of the conditions apply **else** will be invoked i.e. calculate the height by finding the maximum of the heights at each end of the strip. As most strips will not have a turning point, it is important to have the **else** to alleviate unnecessary calculation. If we solve  $u'(x) = 0$  we use the Newton-Raphson method. I have chosen an arbitrary 5 iterations the reader may choose to use more or less iterations.

I also chose the mid point of the strip as the first approximation, i.e.  $a + \frac{(b-a)(2j+1)}{2n}$  hence the solution of the equation is found thus,  $ITERATE \left( x - \frac{\frac{d}{dx} u}{\frac{d^2}{dx^2} u}, x, a + \frac{(b-a)(2j+1)}{2n}, 5 \right)$  will give the  $x$  co-

ordinate of the turning point. This now needs to be substituted into  $u(x)$  to get height of the rectangle. This expression for the height will be

$$\lim_{x \rightarrow ITERATE \left( x - \frac{\frac{d}{dx} u}{\frac{d^2}{dx^2} u}, x, a + \frac{(b-a)(2j+1)}{2n}, 5 \right)} u.$$

It is possible that the turning point is not necessarily the maximum height in the strip, so to be safe use the maximum (or minimum) of the end points and turning point.

The **then** argument will become

$$MAX \left( \lim_{x \rightarrow ITERATE \left( x - \frac{\frac{d}{dx} u}{\frac{d^2}{dx^2} u}, x, a + \frac{(b-a)(2j+1)}{2n}, 5 \right)} u, \lim_{x \rightarrow a + \frac{(b-a)j}{n}} u, \lim_{x \rightarrow a + \frac{(b-a)(j+1)}{n}} u \right).$$

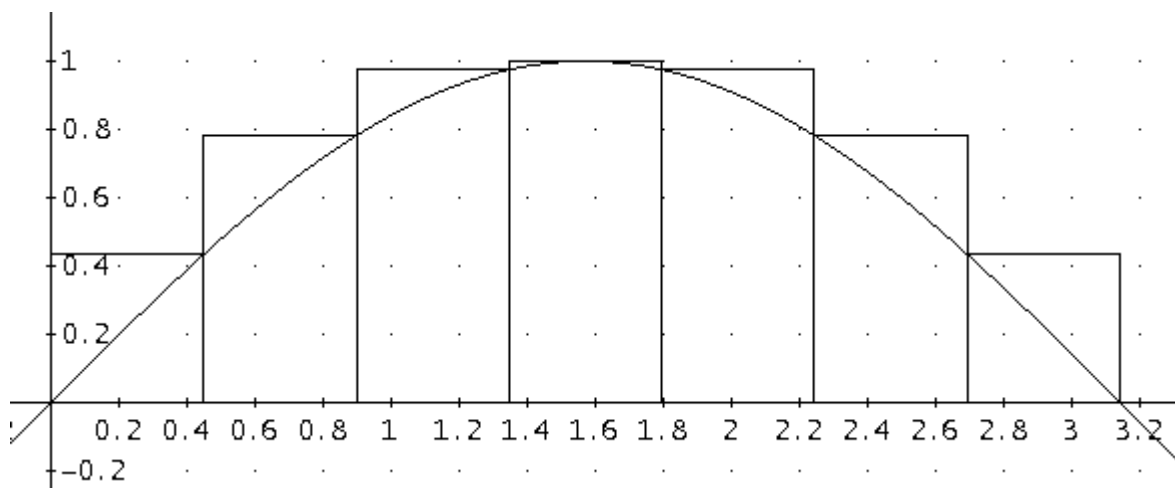
The full functions are displayed how they are edited. I find that I prefer to construct functions within the algebra window using F3 and F4. However, functions of this size cannot be edited on the author line. So I had to employ a line editor and load the file through Transfer Load. However, the debugging routines employed when entering an authored expression do not apply. This makes the debugging process doubly hard! Any suggestions Soft Warehouse!

(This paragraph concerning editing and debugging is – fortunately enough – historic, Josef)

```
riemupsum(u,a,b,n):=VECTOR(rect(a + j·(b - a)/n, 0, IF((LIM(DIF(u, x), x,
a + j·(b - a)/n, 0) < 0 ∧ LIM(DIF(u, x), x, a + (j + 1)·(b - a)/n, 0) >
0) ∨ (LIM(DIF(u, x), x, a + j·(b - a)/n, 0) > 0 ∧ LIM(DIF(u, x), x, a +
(j + 1)·(b - a)/n, 0) < 0), MAX(LIM(u, x, ITERATE(x - DIF(u, x)/DIF(u, x,
2), x, a + (b - a)·(2·j + 1)/(2·n), 5), 0), LIM(u, x, a + (b - a)·j/n),
LIM(u, x, a + (b - a)·(j + 1)/n)), MAX(LIM(u, x, a + (b - a)·j/n), LIM(u,
x, a + (b - a)·(j + 1)/n))), (b - a)/n, j, 0, n - 1)
```

```
riemlosum(u,a,b,n):=VECTOR(rect(a + j·(b - a)/n, 0, IF((LIM(DIF(u, x), x,
a + j·(b - a)/n, 0) < 0 ∧ LIM(DIF(u, x), x, a + (j + 1)·(b - a)/n, 0) >
0) ∨ (LIM(DIF(u, x), x, a + j·(b - a)/n, 0) > 0 ∧ LIM(DIF(u, x), x, a +
(j + 1)·(b - a)/n, 0) < 0), MIN(LIM(u, x, ITERATE(x - DIF(u, x)/DIF(u, x,
2), x, a + (b - a)·(2·j + 1)/(2·n), 5), 0), LIM(u, x, a + (b - a)·j/n),
LIM(u, x, a + (b - a)·(j + 1)/n)), MIN(LIM(u, x, a + (b - a)·j/n), LIM(u,
x, a + (b - a)·(j + 1)/n))), (b - a)/n, j, 0, n - 1)
```

To prove that these functions do work below is a Derive screen dump using these function on  $\sin x$  between 0 and  $\pi$  for 7 rectangles.



DERIVE 6 Screendump from 2005

It is possible that the reader may be able to find a function where the Newton-Raphson method diverges from the solution of  $u'(x) = 0$ , hence not giving the true Riemann rectangles. I would be grateful to receive any such functions or any other correspondence on these routines (in English please) from Derive users.

File RiemTerence.dfw

# True Riemann Rectangles Part 2

*Josef Böhm, Würmla*

At first I tried to make the files from above better “readable”. I followed Ternece’s ideas and used my own work with upper sums and lower sums. I believe that the file RIEMANN2.MTH will explain itself.

1: [a :=, b :=, F(x) :=]

2: 
$$\left[ F1(t) := \lim_{x \rightarrow t} \frac{d}{dx} F(x), D(n) := \frac{b - a}{n} \right]$$

3: [XL(n) := a + (k - 1) · D(n), XR(n) := a + k · D(n), XM(n) := a + (k - 0.5) · D(n)]

4: [FMAX(n) := MAX(F(XL(n)), F(XR(n))), FMIN(n) := MIN(F(XL(n)), F(XR(n)))]

5: RELEXT(n) := 
$$\lim_{x \rightarrow \text{ITERATE}(x - \partial(F(x), x) / \partial(F(x), x, 2), x, XM(n), 5)} F(x)$$

FRMAX(n) :=  
If F1(XL(n)) · F1(XR(n)) < 0  
6: MAX(RELEXT(n), FMAX(n))  
FMAX(n)

FRMIN(n) :=  
If F1(XL(n)) · F1(XR(n)) < 0  
7: MIN(RELEXT(n), FMIN(n))  
FMIN(n)

8: “Uppersum – Obersumme”

9: OSUM(n) := VECTOR 
$$\left( \begin{bmatrix} XL(n) & 0 \\ XL(n) & FRMAX(n) \\ XR(n) & FRMAX(n) \\ XR(n) & 0 \end{bmatrix}, k, 1, n \right)$$

10: “Lowersum – Untersumme”

11: USUM(n) := VECTOR 
$$\left( \begin{bmatrix} XL(n) & 0 \\ XL(n) & FRMIN(n) \\ XR(n) & FRMIN(n) \\ XR(n) & 0 \end{bmatrix}, k, 1, n \right)$$

You have to author the function  $f(x) :=$  and the intervall  $a :=$  and  $b :=$  (Can be done in one vector – between [...], see DNL#7). Then plot OSUM(n) or USUM(n) – with n being the number of rectangles. Don’t forget to set “Approximate before Plotting” in the 2D-Graph Window.

12:  $[F(x) := \sin(8 \cdot x) + 1.5, a := 0, b := 4]$

13:  $[F(x), \text{OSUM}(10)]$

Figure 1 seems to be ok, but ...

14:  $[F(x), \text{OSUM}(5)]$

Figure 2 is obviously no upper sum!

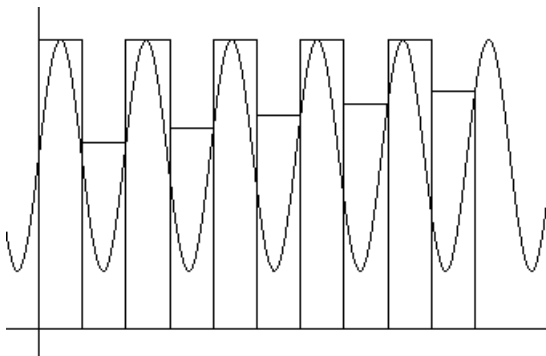


Figure 1

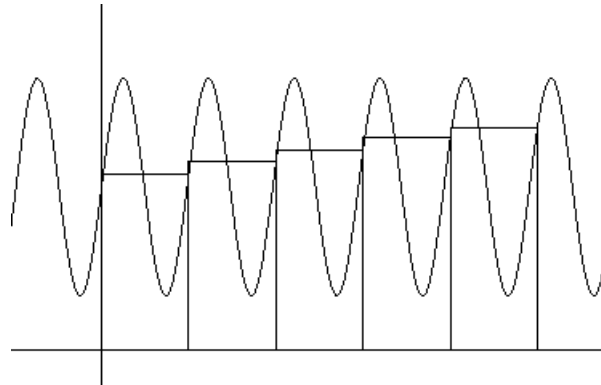


Figure 2

15:  $F(x) := \sin(6 \cdot x) \cdot e^{-0.5 \cdot x} + 1.5$

16:  $[F(x), \text{USUM}(10)]$

Figure 3 is not so bad!

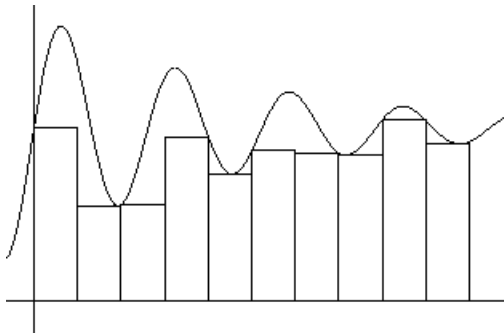


Figure 3

17:  $F(x) := \sin(10 \cdot x) \cdot e^{-0.8 \cdot x} + 1.5$

18:  $[F(x), \text{USUM}(4)]$

Figure 4 is wrong

19:  $[F(x), \text{USUM}(2), \text{OSUM}(2)]$

Figure 5 is wrong

20:  $b := \pi$

21:  $[F(x), \text{USUM}(10), \text{OSUM}(10)]$

but Figure 6 and 7 are correct!

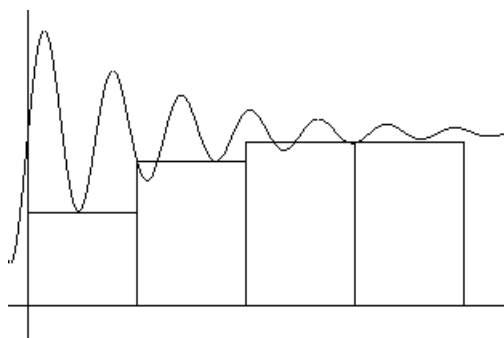


Figure 4

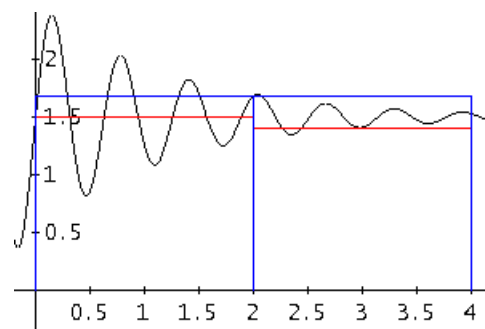


Figure 5

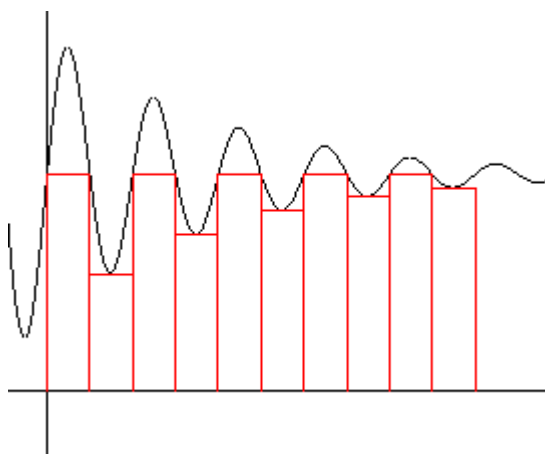


Figure 6

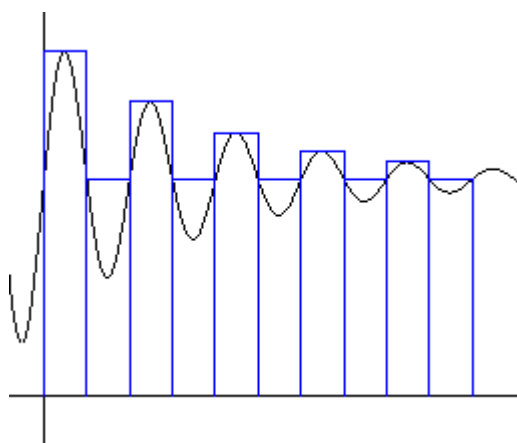


Figure 7

It is obvious that the condition to look for a turning point will fail, if the slopes in the end points of a strip containing such a turning point have the same sign or if there are more extremas within one strip. I produced some plots and sent them to Terence an .... I got an answer which will form the next chapter of our RIEMANN-adventure.

## True Riemann Rectangles Part 3

Terence Etchells, Bolton, UK

Dear Joseph,

Thank you for your letter and screen dumps of my Riemann routines running on those functions. You are quite right, if the width of the rectangle contains more than one turning point then the routine breaks down. This is because only the *end points* are sampled. When one looks at the function the highest (or lowest) point is found by sampling many points in between the end points. You indeed set me a task, but I have solved the problem (it took all day Sunday to write!). Please find enclosed a disk with a new file RIEMANN3.MTH (to save you typing), it is 720 K MSDOS format. The function RIEMUPSUM(u,a,b,n,s) will find the true upper sum rectangles (OBERSUM), the lowersum is easily produced by editing this function.

It differs from the original in that it samples  $s$  points between the end points, if there is a change of sign of  $f'(x)$  between any sample points as the first approximation. The maximum of these points is used. As a consequence the computing times are very long! 60+ seconds on a 386 machine.

```
rect(a,b,m,n):= [[a,b;a,b+m],[a,b+m;a+n,b+m],[a+n,b+m;a+n,b]]
```

```
riemupsum(u,a,b,n,s):= VECTOR(RECT(a+j*(b-a)/n,0,MAX(VECTOR
(IF(LIM(DIF(u,x),x,a+(b-a)*j/n+(a+(b-a)*(j+1)/n-(a+(b-a)*j/n))*k/s)>=0 AND
LIM(DIF(u,x),x,a+(b-a)*j/n+(k+1)*(a+(b-a)*(j+1)/n-(a+(b-a)*j/n))/s)<=0 OR
LIM(DIF(u,x),x,a+(b-a)*j/n+(a+(b-a)*(j+1)/n-(a+(b-a)*j/n))*k/s)<=0 AND
LIM(DIF(u,x),x,a+(b-a)*j/n+(a+(b-a)*(j+1)/n-(a+(b-a)*j/n))*k/s)>=0,
LIM(u,x,ITERATE(x-DIF(u,x)/DIF(u,x,2),x,(a+(b-a)*j/n+(b-a)*(2*k+1)/(2*s*n)),5)),
Max(LIM(u,x,a+(b-a)*j/n),LIM(u,x,a+(b-a)*(j+1)/n))),k,0,s-1)),(b-a)/n,j,0,n-1)
```

You will find this file under the name RIEMANN3.MTH on your disk of the year together with the RIEMLOSUM-function. It was easy to edit and to save as a .MTH file with WORD5.

Having loaded RIEMANN3 in DERIVE I tried to plot the figures from Part 2 once more. See the results (I don't list the functions):

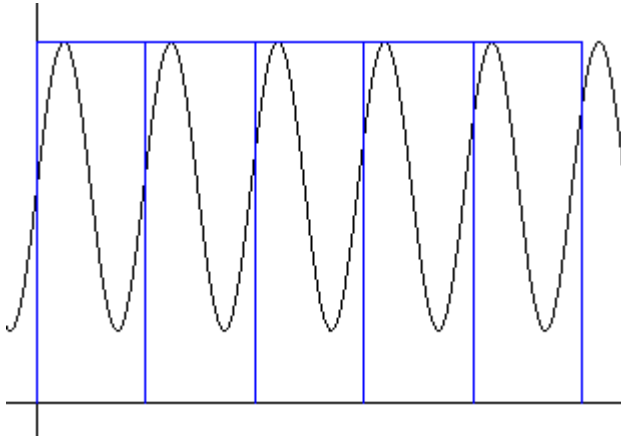
$$\text{RIEMUPSUM}(\text{SIN}(8 \cdot x) + 1.5, 0, 4, 5, 4)$$


Figure 8 (compare with Figure 2)

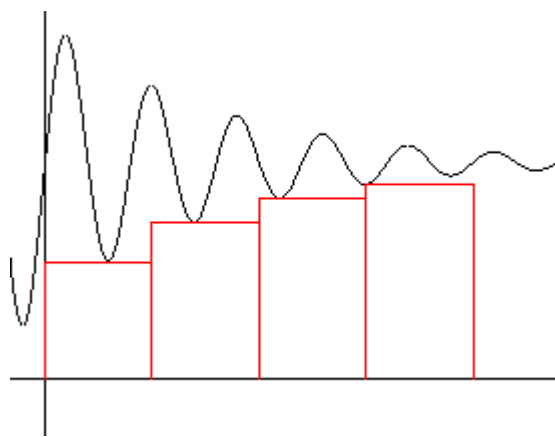
$$\text{RIEMLOSUM}(\text{SIN}(10x)e^{-0.8x} + 1.5, 0, \pi, 4, 3)$$


Figure 9 (compare with Figure 4)

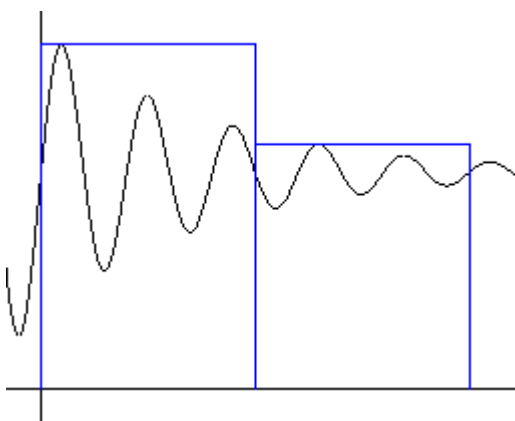
$$\text{RIEMUPSUM}(\text{SIN}(10x)e^{-0.8x} + 1.5, 0, \pi, 2, 5)$$


Figure 10 (compare with Figure 5)

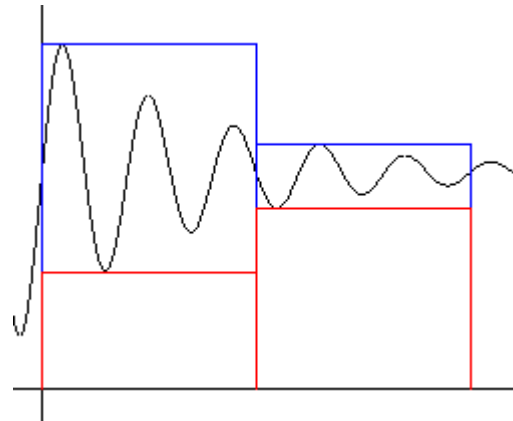
$$\text{RIEMLOSUM}(\text{SIN}(10x)e^{-0.8x} + 1.5, 0, \pi, 2, 5)$$


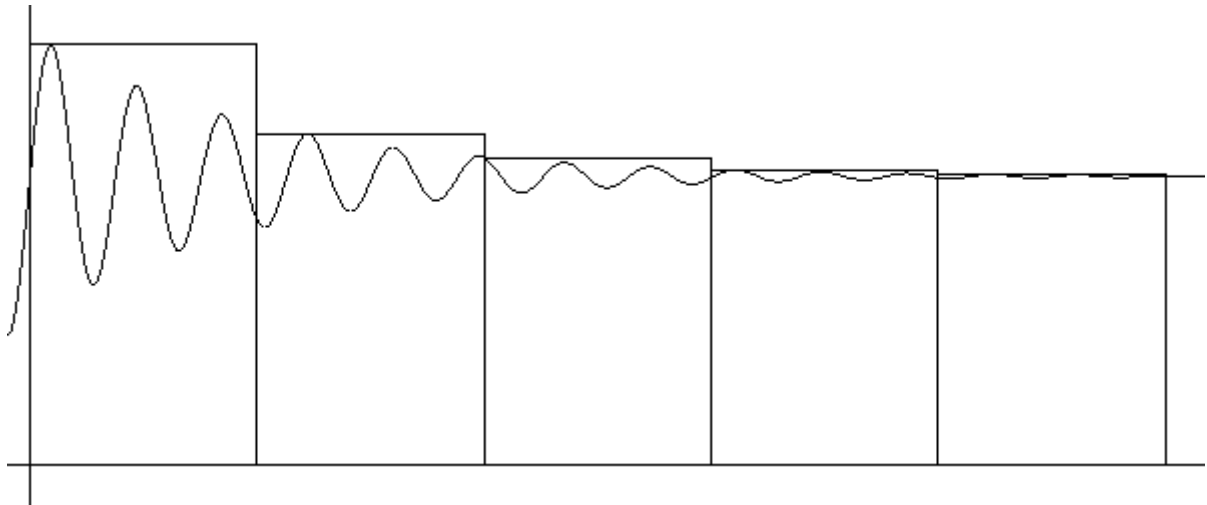
Figure 11 (compare with Figure 5)

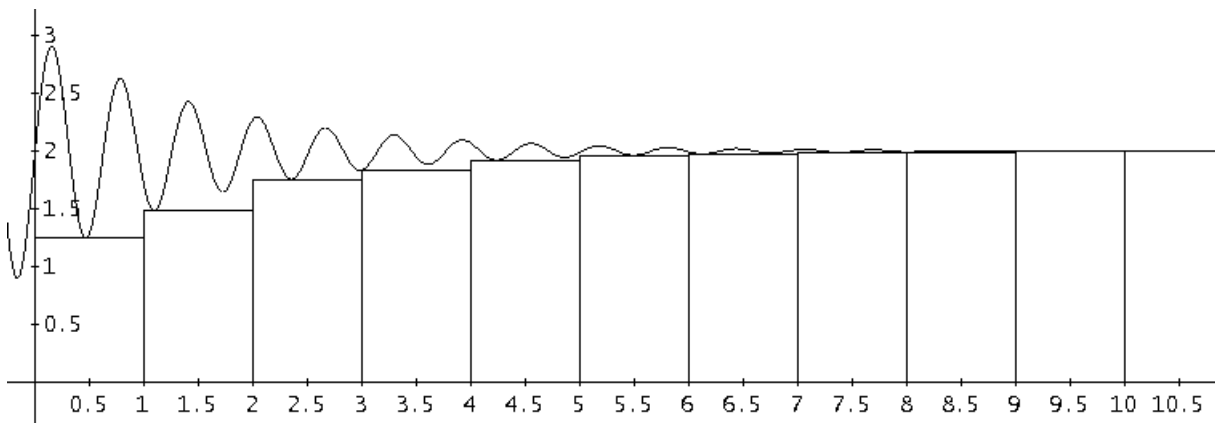
### ***Congratulations, Terence!!***

In times of DERIVE 5 and DERIVE 6 when DERIVE has become programmable, we can try to make such routines better readable and understandable for everybody. I am using an auxiliary function `mysol`, which looks for possible zeros of the 1<sup>st</sup> derivative in the intervals.

```
rupsum(u, a, b, n, s := 0.1, d, u2, i, tp, lh, str) :=
  Prog
    d := (b - a)/n
    u2 := ∂(u, x)
    i := 0
    lh := []
    str := []
  Loop
    If i = n
      RETURN [u, str]
    tp := mysol(u2, x, a + i*d, a + (i + 1)*d, s)
    h := MAX(VECTOR(u, x, APPEND([a + i*d, a + (i + 1)*d], tp)))
    str := APPEND(str, [[a + i*d, 0; a + i*d, h; a + (i + 1)*d, h; a + (i + 1)*d, 0]])
    i := i + 1
```



$$\text{rupsum}(\text{SIN}(10 \cdot x) \cdot e^{-0.6 \cdot x} + 2, 0, 10, 6, 0.2)$$


$$\text{rlosum}(\text{SIN}(10 \cdot x) \cdot e^{-0.6 \cdot x} + 2, 0, 10, 10, 0.2)$$


I have to admit that Terence's routine is much faster than my program. I will try to improve it and also try to produce shaded rectangles.

**Esperimenti didattici con un  
programma di Computer Algebra  
Schede di Lavoro**

Rosamaria Castelletti, Segrate

In the last DNL I gave a summary of an article, first published in "*L'insegnamento della matematica e delle scienze integrate*" by *Centro Ugo Morin, 31010 Paderno del Grappa (Italy)*. Here are some of Mrs Castelletti's worksheets announced in DNL#7:

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TAVOLA 3

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Liceo Scientifico Statale "N. Machiavelli" - Pioltello

a.s. 1989/90

DERIVE

Classe prima

---

Con l'aiuto della grafica risolviamo alcune disequazioni di vario tipo individuando le soluzioni nel piano cartesiano.

Prima di tutto entriamo in Derive e, al solito, suddividiamo lo schermo in due finestre:

a) Consideriamo la disequazione fratta  $\frac{x^2 - x - 6}{x - 1} > 0$

Per risolverla tracciamo il grafico delle due funzioni

N:  $y = x^2 - x - 6$

D:  $y = x - 1$

(conviene cambiare scala: Scale/x-scale = 3, y-scale = 5)

Cercate ora di individuare dal grafico gli intervalli in cui le funzioni sono entrambe positive ( $y > 0$ ) oppure entrambe negative ( $y < 0$ ) perchè in questi intervalli la frazione N/D sarà .....

Quali sono le soluzioni della disequazione? .....

Se volete trovare i punti di intersezione di una curva con l'asse x basta usare il comando solve per l'equazione della curva esaminata cioè quella evidenziata dal riquadro bianco .....

b) Ripete i passi precedenti confratta  $\frac{2x^2 - 3x + 1}{(x - 2)(x - 1)} \leq 0$

Se volete cancellare i grafici precedenti, usate Delete. Se volete lasciare l'ultimo grafico, usate Butlast.

c) Vediamo ora di risolvere un sistema di disequazioni  $\begin{cases} x^2 - 3x - 10 \leq 0 \\ x - 3 > 0 \end{cases}$

Questa volta dobbiamo visualizzare dove le disequazioni sono entrambe verificate ("l'intersezione"), sempre dal grafico delle curve.

Soluzioni: .....

d) Proviamo ora con il sistema  $\begin{cases} x - 2 > 0 \\ x^2 - 9x + 20 > 0 \\ x^2 - 6x - 7 \leq 0 \end{cases}$

Soluzioni: .....

e) Risolviamo infine, sempre tracciando i grafici, una disequazione irrazionale:

$$2x < 3\sqrt{4x^2 - 13x + 3}.$$

Come prime tracciate i grafici delle due funzioni (per la radice potete usare  $(....)^{(1/2)}$  oppure  $SQRT(....)$ )

Soluzioni: .....

#### TAVOLA 4

Liceo Scientifico Statale "N. Machiavelli" - Pioltello

a.s. 1989/90

DERIVE

Classe prima

1) Entrate in Derive e tracciate il grafico della retta  $y = \frac{1}{2}x + 5$

Il punto di intersezione con l'asse  $x$  è A( ..... )

Il punto di intersezione con l'asse  $y$  è P( ..... )

2) Tracciate il grafico della retta  $y = -2x + 5$ , sullo stesso piano della prima retta.

Il punto di intersezione con l'asse  $x$  è B( ..... )

Il punto di intersezione con l'asse  $y$  è .....

3) Dimostrare che il triangolo ABP è rettangolo

(usare l'inverso del teorema di Pitagora e la formula per calcolare la distanza di due punti)

4) Possiamo affermare che "condizione necessaria e sufficiente affinché due rette siano perpendicolari è che il prodotto dei loro coefficienti angolari sia uguale a -1"?

#### TAVOLA 5

Liceo Scientifico Statale "N. Machiavelli" - Pioltello

a.s. 1989/90

DERIVE

Classe prima

a)  $y = 3kx + 7$

Incominciamo a considerare alcuni casi particolari. Sostituiamo a  $k$  il valore 1 e tracciamo il grafico della curva. Ripetiamo assegnando a  $k$  i valori 2 e 3.

E' giusta l'affermazione "l'equazione  $y = 3kx + 7$  rappresenta un fascio proprio di rette"? .....

Quale è il centro di tale fascio? .....

b) Consideriamo ora l'equazione  $y = 3kx + 1$  (1)

Se  $y = 0$ , si ottiene l'equazione di primo grado ..... (2)

Le intersezioni con l'asse  $x$  dell'equazione (1) sono le soluzioni della (2) ed il loro valore dipenderà dai valori attribuiti al parametro  $k$ .

<b>p26</b>	<b>Rosamaria Castelletti: Worksheets – Schede di Lavoro</b>	<b>D-N-L#8</b>
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Usando il programma Derive notiamo infatti che

per  $k = 0$  si ha  $y = \dots$  il cui grafico è .....

per  $k = 2$  si ha  $y = \dots$  il cui grafico è .....

per  $k = -1/3$  si ha  $y = \dots$  il cui grafico è .....

Posiamo pertanto concludere che l'equazione (1) rappresenta un ..... Le rette passano tutte per il punto I(.....).

Ad ogni rette del fascio è associato un particolare punto di intersezione con l'asse delle  $x$  .....

Tale punto ha ascissa ..... ed ordinata.

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#### TAVOLA 2

Liceo Scientifico Statale "N. Machiavelli" - Pioltello

a.s. 1989/90

DERIVE

Classe seconda

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**OBIETTIVI** – scoprire le regole di calcolo con i radicali aritmetici

– scoprire l'equivalenza fra rappresentazioni con le radici e con gli esponenti razionali

---

1) Immettere  $(x-2)^{1/2}$

Osservazioni: .....

2) Immettere  $[(x-2)^2]^{1/2}$

Osservazioni: .....

3) Sotto quali condizioni sono vere le seguenti uguaglianze e perchè?

$$(x-2)\sqrt{x-2} = \sqrt{(x-2)^2(x-2)} = \sqrt{(x-2)^3} = |x-2|\sqrt{x-2} = (x-2)\sqrt{x-2}$$

4) Quali condizioni devi porre per la validità dei seguenti passaggi? Mancano dei valori assoluti?

$$(x-3)\sqrt[3]{\frac{x-2}{x-3}} = \sqrt[3]{(x-3)^3} \sqrt[3]{\frac{x-2}{x-3}} = \sqrt[3]{\frac{(x-3)^3(x-2)}{x-3}} = \sqrt[3]{(x-3)^2(x-2)}$$

$$\sqrt{(x-3)^2} = |x-3|$$

$$\sqrt[3]{(x-3)^3} = x-3$$

$$\sqrt{(x-3)^3} = (x-3)\sqrt{x-3}$$

## TAVOLA 7

Liceo Scientifico Statale "N. Machiavelli" - Pioltello

a.s. 1989/90

DERIVE

Classe terza

**OBIETTIVO:** scoprire le proprietà di alcune funzione e curve

LE RAPPRESENTAZIONI GRAFICHE DEVONO ESSERE TUTTE RICOSTRUITE SUL FOGLIO DI LAVORO.

INDICARE CON PRECISIONE I PUNTI NOTEVOLI

Per gli esercizi 1-5 scegliere il valore 5 nell'opzione Scale prima di tracciare i grafici.

- 1) Tracciare sullo stesso piano cartesiano le rappresentazioni grafiche di

$$y = \sqrt{9 - x^2} \quad \text{e} \quad y = -\sqrt{9 - x^2}$$

OSSERVAZIONI:

.....

- 2) Considerare il fascio di rette di equazione

$$y = k(x - 5)$$

Evidenziare il parametro  $k$ . Usare il comando MANAGE SUBSTITUTE. Indicare  $3/4$  come valore da sostituire a  $k$ . Ripetere indicando il valore  $-3/4$ . Tracciare le due rette così ottenute nello stesso piano che contiene le curve del punto 1.

OSSERVAZIONI:

.....

- 3) Cosa succede se
- $k$
- è minore di
- $-3/4$
- oppure
- $k$
- è maggiore di
- $3/4$
- ?

E se  $k$  è interno all'intervallo  $(-3/4, 3/4)$ ?Esplorare cosa succede sostituendo a  $k$  diversi valori (comando MANAGE SUBSTITUTE).

Riportare sul foglio di lavoro il grafico ottenuto, scrivendo accanto ad ogni retta il valore di  $k$  che l'ha generata.

.....

- 4) Tracciare la rappresentazione grafica di

$$y = \sqrt{3x - x^2} \quad \text{e} \quad y = -\sqrt{3x - x^2}$$

La curva complessiva ottenuta è .....

Ripeti il lavoro dei punti 1-3 con il fascio di rette di equazione  $y = k\left(x - \frac{3}{2}\right)$ .

OSSERVAZIONI: .....

- 5) Tracciare la rappresentazione grafica di

$$y = -2 + \sqrt{4 - x^2} \quad \text{e} \quad y = -2 - \sqrt{4 - x^2}$$

OSSERVAZIONI: .....

<b>p28</b>	<b>Rosamaria Castelletti: Worksheets – Schede di Lavoro</b>	<b><i>D-N-L#8</i></b>
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- 6) Chiudere la finestra grafica. Considerare la funzione di equazione

$$y = \frac{3}{2} \sin(x) + 2 \cos(x) - \frac{1}{2}$$

Tracciare la rappresentazione grafica della funzione (dopo aver riportato a 1 i valori nell'opzione Scale).

La funzione è periodica? Se sì, quale potrebbe essere il periodo?

La funzione ha punti di massimo e di minimo? Se sì, sapresti determinare con Derive il valore minimo e il valore massimo?

Descrivi il procedimento che hai usato.

.....

.....

.....

- 7) Usate Derive per tracciare il grafico della funzione

$$y = x \sin(x)$$

Tracciare sullo stesso piano le rette  $y = x$  e  $y = -x$ .

OSSERVAZIONI: .....

- 8) Tracciate ora la rappresentazione grafica di

$$y = x^2 \sin(x)$$

Nell'esercizio 7 abbiamo tracciato le rette  $y = x$  e  $y = -x$ . In questo caso quali curve conviene tracciare?

OSSERVAZIONI: .....

.....

- 9) Esplorate infine la seguente funzione:

$$y = x + \sin(x)$$

OSSERVAZIONI: .....

.....

*This is a letter too long to be a part of the DERIVE USER FORUM. It is a lucky chance that I had some ideas in the same direction when I worked through the messages of the Bulletin Board Service. This might be an answer and completion to Mr Robin's letter (Editor). Read now Mr Robin's letter:*

I was interested to read in DNL#7, in the Bulletin Board Section (incidentally how does one access this facility?) the correspondence between Andromeda and Boom-Boom concerning the Normal probability integral. (DNL#7, page 7)

Using DERIVE in Exact Mode NORMAL(1) simplifies to:

$$\text{NORMAL}(1) = \frac{\text{ERF}\left(\frac{\sqrt{2}}{2}\right) + 1}{2}$$

although it takes 9 seconds to do so. To do inverse problems, the equation say  $\text{NORMAL}(x) = 0.95$  is solved in Approximate Mode to give 1.64489, taking 17 seconds.

I came across the approximation given for  $Q(x)$  in the manual for the Casio Programmable Graphical calculator fx7000G. Whilst this was useful and quick to use (instantaneous), I found that I could not dispose the Normal tables unless I also had an inverse for this function. This involves solving the equation  $Q(x) = a$  for values of  $a$  less than 0.5. Newton-Raphson method was very suitable, as  $Q'(x)$  is of course known, but it was necessary to have a good starting value, for the method to work efficiently. To do this I found a simple approximation for  $Q(x)$  namely;

$$\text{EXP}(-0.43 - 0.45(x + 0.78)^2)$$

Now this can easily be inversed to give:

$$x = \text{INVQ}(a) = \sqrt{((- \text{LN}(a) - 0.43)/0.45)} - 0.78$$

which gives us a good starting point for the Newton-Raphson procedure. The direct program for finding  $Q(x)$  is put into a subroutine. The fx7000G takes about 2 seconds to find  $\text{INVQ}(0.05) = 1.644854019$ . I am enclosing copies of the calculator programs for finding  $Q$  (prog 7) and  $\text{INVQ}$  (prog 9). I am sure any reader proficient with programming can easily convert for use in any computer or programmable calculator. I am also enclosing copies of graphs showing the absolute and fractional error in using this crude quadratic approximation.

Yours sincerely

You can find informations about the Bulletin Board in your DERIVE manual, look under Additional Resources.

I don't know which computer you are using, but when I approximate  $\text{NORMAL}(1)$  it only needs 0.1 seconds and for solving  $\text{NORMAL}(x) = 0.95$  my machine needs 0.8 seconds. I can't copy the programs and graphs mentioned above, because of their print quality, and I don't have the special calculator. Maybe Mr. Robin will send a better copy. Thank you. It might be that the following routine will suit you (and other readers) more. Please read a part of a letter to D.Stoutemyer concerning the same problem:

..... I've found the approximating algorithm for the inverse distribution function. It might be interesting for some users.

Approximation of the Normal Distribution function  
 Abramowitz/Stegun: Handbook of Mathematical Functions  
 Dover, New York 1965

Wahrscheinlichkeitsrechnung, Statistik  
 Dietmar Herrman  
 Vieweg, Braunschweig 1983

#1: [p := 0.23164, a := 0.31938, b := -0.35655, c := 1.7814]

#2: [d := -1.8212, e := 1.3302]

#3:  $T(x) := \frac{1}{1 + p \cdot |x|}$

#4:  $F(x) := \frac{1}{\sqrt{2 \cdot \pi}} \cdot e^{-x^2/2}$

#5:  $\Phi(x) := 1 - F(x) \cdot (a \cdot T(x) + b \cdot T(x)^2 + c \cdot T(x)^3 + d \cdot T(x)^4 + e \cdot T(x)^5)$

#6:  $\text{DIST}(x) :=$   
 If  $x > 0$   
 $\Phi(x)$   
 $1 - \Phi(x)$

The inverse Distribution Function:

#7:  $R(w) := \sqrt{\text{LN}\left(\frac{1}{(1-w)^2}\right)}$

#8: [c2 := 0.010328, c1 := 0.80285, c0 := 2.5155]

#9: [d1 := 1.4327, d2 := 0.18926, d3 := 0.001308]

#10:  $Z(w) := c2 \cdot R(w)^2 + c1 \cdot R(w) + c0$

#11:  $N(w) := d3 \cdot R(w)^3 + d2 \cdot R(w)^2 + d1 \cdot R(w) + 1$

#12:  $\text{INV}_\Phi(w) := R(w) - \frac{Z(w)}{N(w)}$

#13:  $\text{INV\_DIST}(w) :=$   
 If  $w > 0.5$   
 $\text{INV}_\Phi(w)$   
 $- \text{INV}_\Phi(1 - w)$

#14: [DIST(0), DIST(1), DIST(2.97), DIST(-1.5), DIST(-1.5) + DIST(1.5)]

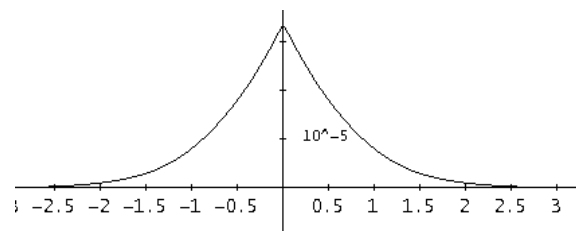
#15: [0.4999, 0.8413, 0.9985, 0.06680, 1]

#16: [INV\_DIST(0.9985), INV\_DIST(0.0668)]

#17: [2.968, -1.500]

Plotting the absolute error:

|NORMAL(x) - DIST(x)|





Here are some graphics from Mr Setif. In the first functions he demonstrates the Polar Coordinates: (Set coordinate system to Polar!)

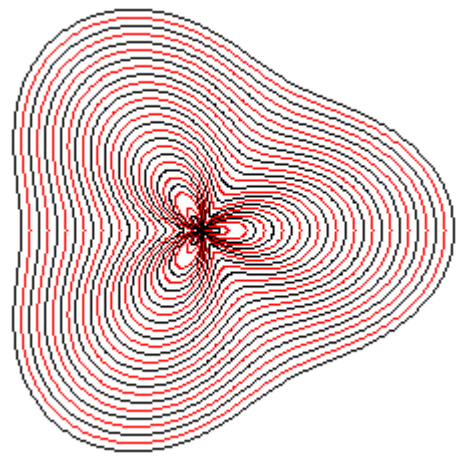
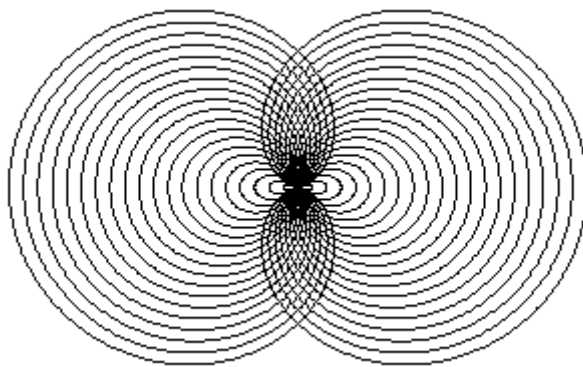
picture 1

$$\left[ \text{VECTOR} \left( 2 \cdot \cos(x) + \frac{k}{5}, k, 0, 10 \right), \text{VECTOR} \left( -2 \cdot \cos(x) + \frac{k}{5}, k, 0, 10 \right) \right]$$

picture 2; #5 in black and #6 in red

$$\text{VECTOR} \left( \frac{\cos(3 \cdot x)}{2} + \frac{k}{4}, k, 12 \right)$$

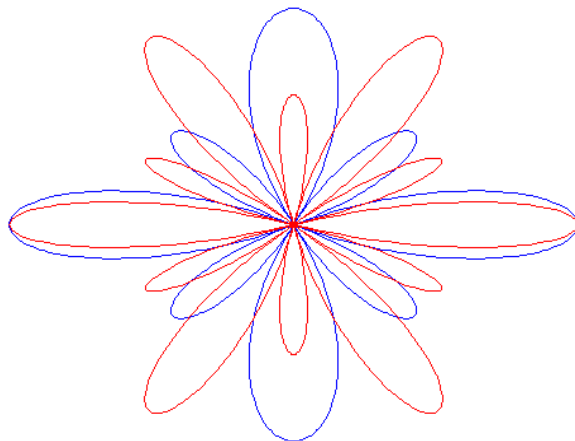
$$\text{VECTOR} \left( \frac{\cos(3 \cdot x)}{2} + \frac{k}{4}, k, 0.5, 11.5 \right)$$



picture 3

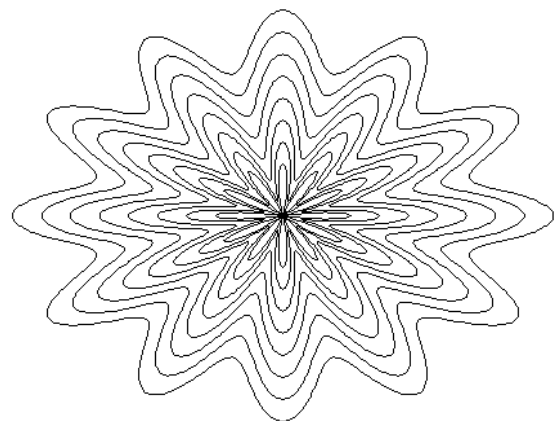
$$2 \cdot \cos(4 \cdot x) + 0.5$$

$$2 \cdot \cos(6 \cdot x) + 0.5$$



picture 4

$$\text{VECTOR} \left( \frac{\cos(12 \cdot x)}{3} + \frac{k}{4}, k, 1, 8 \right)$$



picture 5

$$2 \cdot \cos\left(\frac{5 \cdot x}{2}\right) + 0.5$$

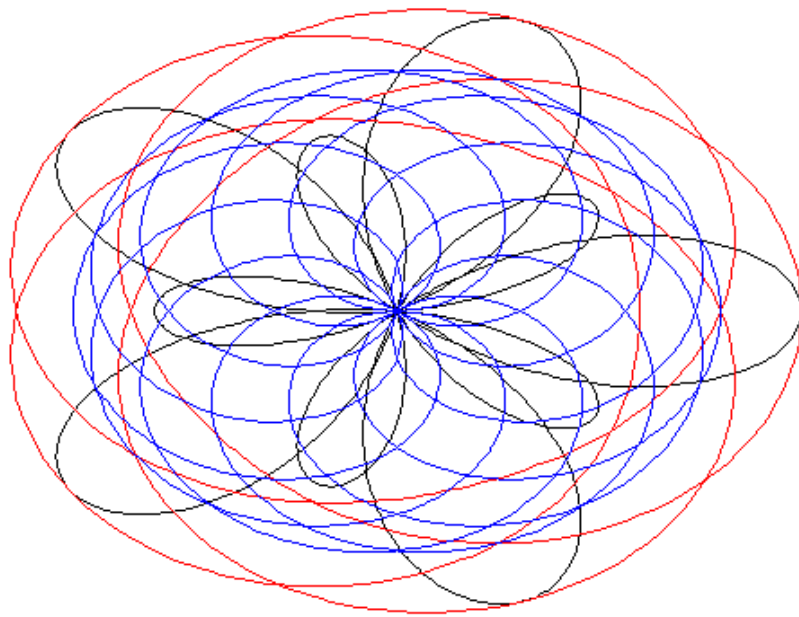
$$0 \leq x \leq 4\pi$$

$$\frac{\cos\left(\frac{5 \cdot x}{4}\right)}{2} + 2$$

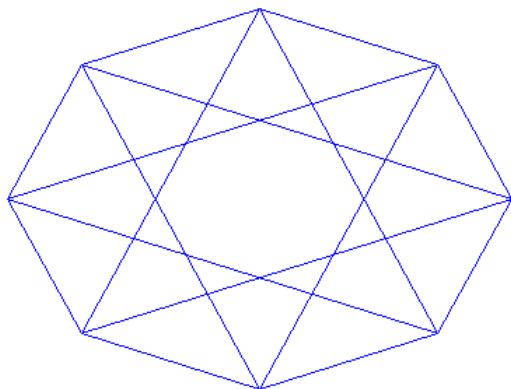
$$0 \leq x \leq 8\pi$$

$$2 \cdot \cos\left(\frac{5 \cdot x}{8}\right)$$

$$0 \leq x \leq 16\pi$$



Change the coordinate system to Rectangular

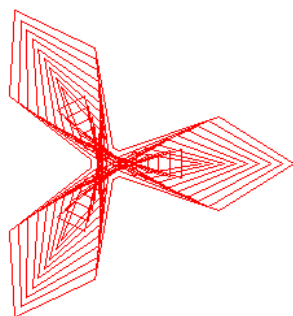
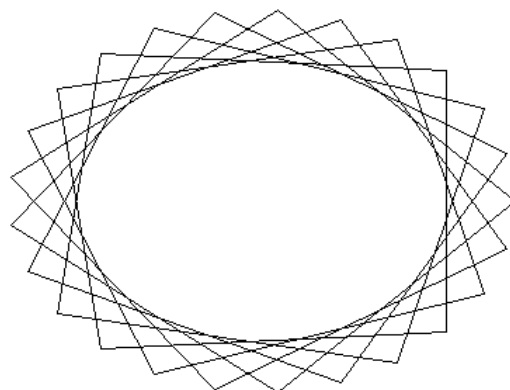
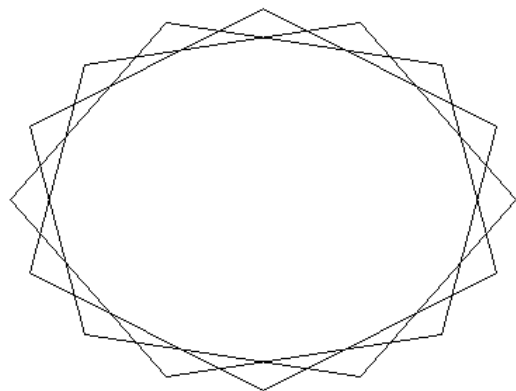


setif\_6

[POLYG(2, x, 0, 2·π, 8), POLYG(2, x, 0, 6·π, 8)]

POLYG(2, x, 0, 6·π, 16)

POLYG(2, x, 0, 12·π, 25)



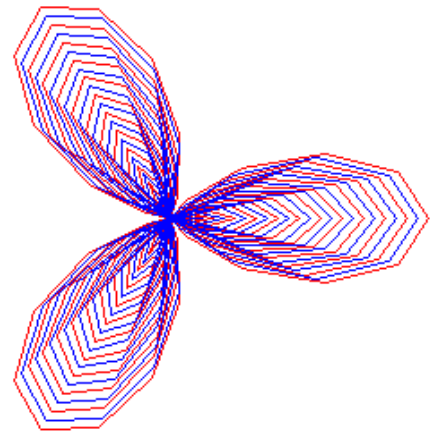
setif\_7

VECTOR(POLYG(2·cos(3·x) +  $\frac{k}{4}$ , x, 0, 2·π, 15), k, 0, 8)

setif\_8

$$\text{VECTOR}\left(\text{POLYG}\left(2\cdot\cos(3\cdot x) + \frac{k}{3}, x, 0, 2\cdot\pi, 30\right), k, 0, 6\right)$$

$$\text{VECTOR}\left(\text{POLYG}\left(2\cdot\cos(3\cdot x) + \frac{k}{3}, x, 0, 2\cdot\pi, 30\right), k, \frac{1}{2}, 6\right)$$



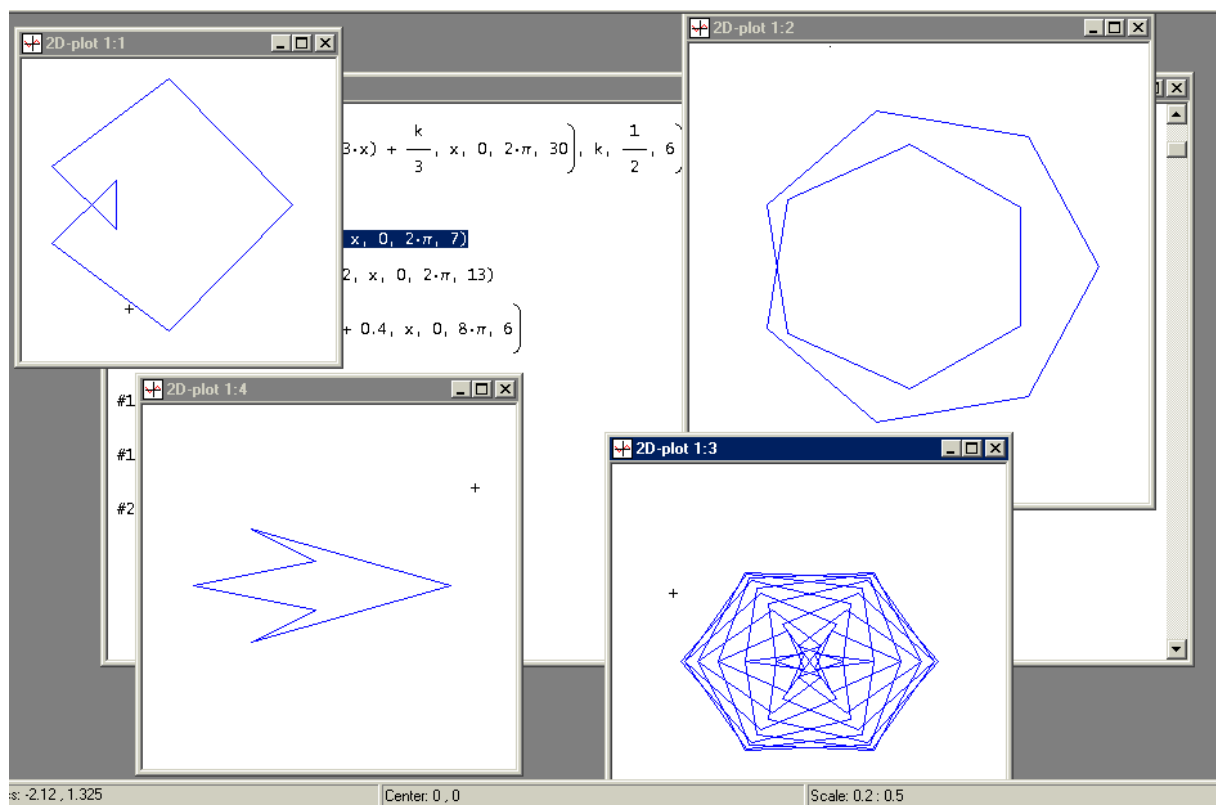
setif\_9

$$\text{POLYG}(2\cdot\cos(x) + 1, x, 0, 2\cdot\pi, 7)$$

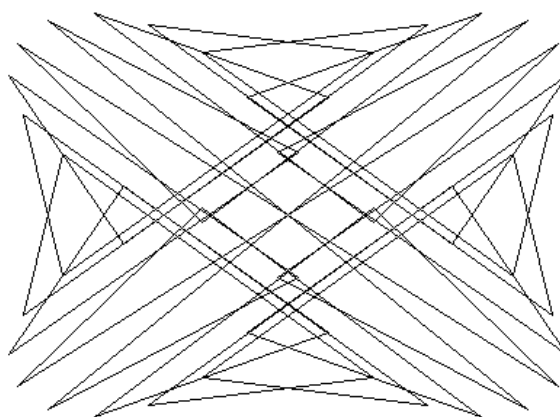
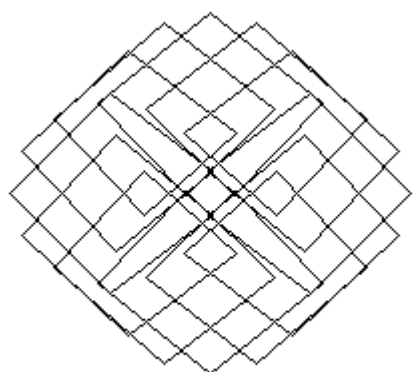
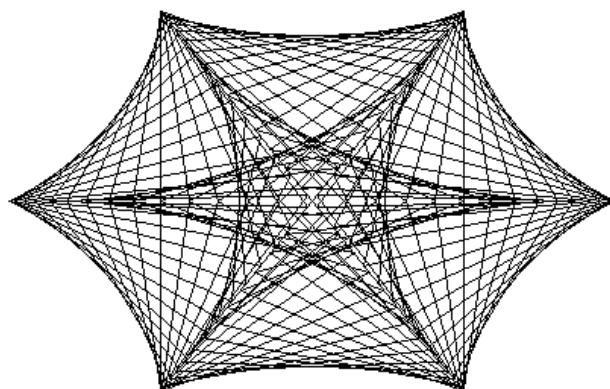
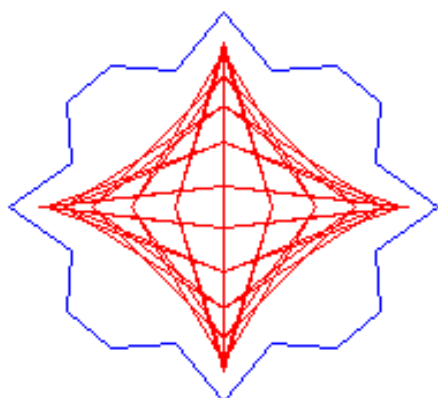
$$\text{POLYG}(2\cdot\cos(x) + 0.2, x, 0, 2\cdot\pi, 13)$$

$$\text{POLYG}\left(2\cdot\cos\left(\frac{5\cdot x}{4}\right) + 0.4, x, 0, 8\cdot\pi, 6\right)$$

$$\text{POLYG}\left(2\cdot\cos\left(\frac{5\cdot x}{12}\right), x, 0, 24\cdot\pi, 72\right)$$



See some more pictures on the next page.



setif\_10

$$\text{POLYG}\left(\frac{\cos(12 \cdot x)}{3} + 2, x, 0, 2 \cdot \pi, 20\right)$$

$$\text{POLYG}\left(2 \cdot \cos\left(\frac{13 \cdot x}{12}\right), x, 0, 24 \cdot \pi, 48\right)$$

$$\text{POLYG}\left(2 \cdot \cos\left(\frac{29 \cdot x}{30}\right), x, 0, 60 \cdot \pi, 180\right)$$

$$\text{POLYG}\left(2.5 \cdot \cos\left(\frac{6 \cdot x}{7}\right), x, 0, 14 \cdot \pi, 52\right)$$

$$\text{POLYG}\left(5 \cdot \cos\left(\frac{6 \cdot x}{7}\right) \cdot \sin\left(\frac{6 \cdot x}{7}\right), x, 0, 14 \cdot \pi, 52\right)$$

$$\text{POLYG}\left(6 \cdot \cos\left(\frac{6 \cdot x}{7}\right) \cdot \sin\left(\frac{6 \cdot x}{7}\right), x, 0, 14 \cdot \pi, 78\right)$$

$$\text{POLYG}\left(6 \cdot \cos\left(\frac{6 \cdot x}{7}\right) \cdot \sin\left(\frac{6 \cdot x}{7}\right)^2, x, 0, 14 \cdot \pi, 78\right)$$

$$\text{POLYG}\left(\frac{6 \cdot \cos\left(\frac{6 \cdot x}{7}\right)}{\sin\left(\frac{6 \cdot x}{7}\right)}, x, 0, 14 \cdot \pi, 78\right)$$

