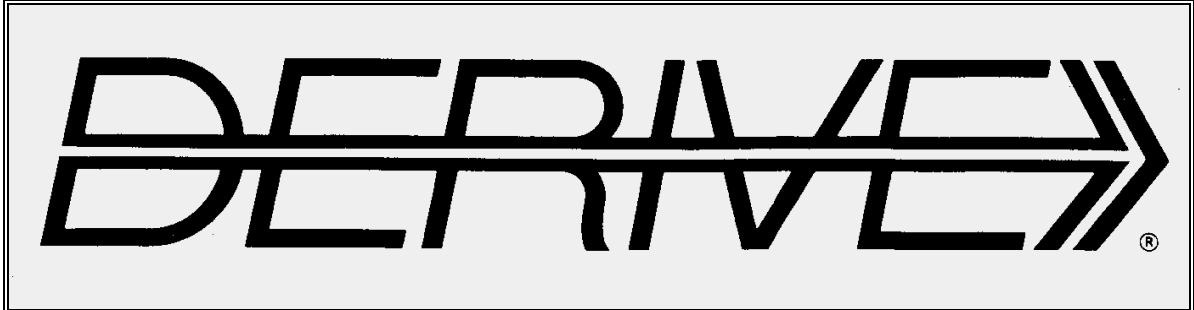


**THE BULLETIN OF THE**



**USER GROUP**

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- [1] **DERIVE im Mathematik- und Physikunterricht**, Günter Scheu  
F. Dümmeler Verlag, Bonn, Dümmelerbuch 4592, 1994
- [2] **Mathematik am PC, Einführung in DERIVE**, Bernhard Kutzler  
Soft Warehouse – Hagenberg, 1994
- [3] **Mathematics on the PC**, Bernhard Kutzler  
Soft Warehouse – Hagenberg, 1994

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### **First North American DERIVE User Group Meeting**

Sunday, November 20, 1994, 9:00 – 13:00  
Room Europe 10, Walt Disney World Dolphin  
Lake Buena Vista, Florida, USA

Report by Bernhard Kutzler

The first North American DERIVE User Group Meeting was organized in the frame of the International Conference on Teaching Collegiate Mathematics (ICTCM) Nov. 17-20, 1994. Despite the early Sunday morning time at the end of the conference and despite of the temptation of being in the middle of Disney World with its many fun parks, approximately 30 people attended the meeting, among them Bert Waits, one of the two founders of the ICTCM conference series, several delegates from Europe and one from Australia.

I had the great honor of chairing the meeting and started by reading Josef's welcome note. It was only half as charming as if Josef would have spoken himself, but people told me that just by hearing his words they almost could see him there. What a marvellous compliment from those who know Josef personally.

David Stoutemyer, one of the two fathers of DERIVE, was the first speaker. He gave some insight in DERIVE version 3. His lecture was followed by presentations of Jeanette Palmiter (USA), David Sjöstrand (Sweden), Lisa Townsley-Kulich (USA), Terence Etchells (UK), Robert Mayes (USA) and Bernhard Kutzler (Austria).

We had many nice discussions following the lectures, met with new DERIVE friends, and once again enjoyed sampling Hawaiian macadamia nuts (which were sponsored by Soft Warehouse, Inc.). It was a very nice start for more North American DERIVE activities.

---

Bernhard, many thanks for your nice report. I am jealous, not because of Disney World, but because of the macadamia nuts. There is a challenge for us in Europe. Dear German members (and members from the countries in the neighbourhood. I'm glad to announce the

### **2. Deutsches DERIVE User Group Treffen im Rahmen der MNU 1995, Nürnberg (Ostern 1995) Mittwoch, 12. 4. 1995 14.00 – 16.00 Uhr**

Gäste und Interessierte sind natürlich herzlich willkommen. Falls Sie den Wunsch haben, einen Kurzvortrag zu halten, uns etwas vorzuzeigen, Wünsche oder Anregungen darzulegen, wäre ich für eine Vorinformation sehr dankbar. Ich hoffe sehr, mit Ihrer Hilfe ein kurzes Programm zusammenstellen zu können. Ich danke jetzt schon Herrn Wolfgang Pröpfer für die Organisation des Treffens. Näheres erfahren Sie im nächsten DNL (März 1995).

Liebe DUG-Mitglieder,

Anfangs dachte ich, es würde schwierig werden, für jeden DNL einen "Letter of the Editor" zu schreiben. Ganz im Gegenteil, ich habe immer Probleme, all das, was ich Ihnen so nebenbei noch mitteilen will in einer Spalte unterzubringen. In diesem DNL möchte ich Sie auf die bereits angekündigte Utility von Sergey Biryukow hinweisen. Die zahlreichen Files finden Sie ebenso wie die beiden Demos im Unterverzeichnis <LABEL>. Es muß eine ungeheure Arbeit gewesen sein, die Zeichensätze zu entwerfen. Sergey und sein Team bitte vor den Vorhang.

Für mich ist es besonders erfreulich feststellen zu können, daß der DNL vielfach nicht nur rasch durchgeblättert wird, um spezielle „Rosinen“ für seinen Bedarf zu finden, sondern, daß sich viele Leser ernsthaft mit den angebotenen Themen auseinandersetzen. Ein Ergebnis finden Sie im User Forum, zwei weitere werden erwähnt. Sie sind so umfangreich, daß sie einen eigenen Artikel wert sind.

Nun muß ich ein Versäumnis nachholen. Leider habe ich bisher vergessen G. Scheu's zweites DERIVE Buch vorzustellen. Das wird jetzt gerne mit einer Bitte um Entschuldigung nachgeholt. Ich möchte besonders darauf hinweisen, daß hier zahlreiche physikalische Anwendungen behandelt werden. Zufrieden, Günter?

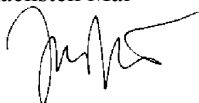
Beachten Sie bitte die beiliegenden Informationen. Beim DERIVE Journal möchte ich nochmals daran erinnern, daß Sie als DUG-Mitglied einen beträchtlichen Preisvorteil genießen.

Nun habe ich eine gute und eine schlechte Nachricht für Sie. Zuerst die schlechte: wir werden den DUG-Mitgliedsbeitrag erhöhen, und zwar auf S 340.- für Österreich, DM 52.-, bzw. BP 22.- für Europa und US\$ 40.- für Übersee. Aber jetzt die (hoffentlich) gute: der Umfang des DNL wird überproportional auf mindestens 40 Seiten erweitert. Fassen Sie das auch als Bitte um weitere Beiträge auf (Sie sind aber recht fleißig!!). Kein Beitrag geht verloren, und wenn der eine oder andere längere Zeit nicht erscheint, dann hat dies ausschließlich technische Ursachen (Platzbedarf, Themenzugehörigkeit, Übersetzung, ....). Für 1995 habe ich eine Ausgabe mit Schwerpunkt Geometrie vorgesehen, eine andere soll sich mit dem Geschehen in österreichischen Schulklassen beschäftigen.

Beachten Sie bitte die beiliegende Rechnung für den Mitgliedsbeitrag 1995. Sie erhalten automatisch ein Quittung mit dem DNL#18.

Ich wünsche Ihnen ein frohes Fest und ein erfolgreiches Neues Jahr 1995.

Mit den besten Grüßen bis zum nächsten Mal



Dear DUG Members,

Producing the first issues of the DNL I thought that it would be difficult or impossible to write a "Letter of the Editor" for each DNL. On the contrary, I've problems to bring all the additional facts I want to tell you into one column. In this DNL I'd like to draw your attention to Sergey Biryukov's utility. You can find the numerous files together with two demos in the folder <LABEL>. It must have been an enormous work to design and edit the font sets. Sergey and his team, in front of the curtain!

It is a great pleasure for me to see that many of you do not only browse the DNL to find special "raisins" for their own purpose but are dealing seriously with the offered items. You can find one result in the User Forum, two others are so voluminous that they are worth an own contribution.

I must admit that I have to make good something. I don't know why, but I have forgotten to place Günter Scheu's 2<sup>nd</sup> DERIVE book on our Book Shelf until now. I want to point especially to the fact that a couple of physics applications can be found in this book. Satisfied Günter?

Please take notice of the information enclosed. Concerning the DERIVE Journal I would like to remind you that as a DUG member you benefit from a considerable reduction.

Now I have one good and one bad news for you. Let me take the bad one first: We will raise the DUG Membership fee up to AS 340 for Austria, DM 52 or BP 22 for Europe and US\$ 40 for other destinations. But here is the – hopefully – good news: the contents of the DNL will be extended overproportionally to at least 40 pages (instead of 30). Please take this fact as a call for further contributions. (I must confirm that you have been very active and reliable until now.)

No contribution – large or small – is getting lost. If one or another will not be published for a longer time then this is caused only by technical reasons (amount of space, subject not suited for the issue, translation, ...). For 1995 I've in mind to dedicate one issue to geometry and another one shall deal mainly with activities in Austrian classrooms.

Please pay attention to the enclosed invoice for the renewal of your membership. You will automatically receive a receipt together with DNL#18.

My wife Noor and I wish you all personally known or not a Merry Christmas and a successful New Year 1995.

Sincerely yours



The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE User Group*. It is published at least four times a year with a contents of 30 pages minimum. The goals of the *D-N-L* are to enable the exchange of experiences made with *DERIVE* as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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D'Lust 1  
Austria  
Phone: 43-(0)2275/8207

### **Contributions:**

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *D-N-L*. It must be said, though, that non-English articles are very welcome nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in *D-N-L*. The more contributions you will send, the more lively and richer in contents the *DERIVE Newsletter* will be.

### **Preview: (Contributions for the next issues):**

Stability of systems of ODEs, Kozubik, SLO  
Prime Iterating Number Generators, Wild, UK  
Graphic Integration, Probability Theory, Linear Programming, Böhm, AUS  
DERIVE in Austrian Schools, some examples, Lechner, Voigt, Eisler a.o., AUS  
Tilgung fremderregter Schwingungen, Klingen, GER  
Continued Fractions and the Bessel Functions, Cordobá a.o., ESP  
Turtle Commands in DERIVE, Lechner, AUS  
DREIECK.MTH, Wadsack, AUS  
IMP Logo and Misguided Missiles, Sawada, HAWAII  
"Reverse" Discussion of Curves, Reichel, AUS  
Reichel - Klingen - Böhm - Splines, A triathlon, AUS & GER  
3D Geometry, Reichel, AUS  
Parallel- and Central Projection, Böhm, AUS  
Conic Sections, Fuchs, AUS  
A Trick for Plotting 2D Plots, Roanes & Roanes, ESP  
Setif, France; Vermeylen, Belgium; Leinbach, USA  
Lymer, FRA; Baum, GER; Kayser, GER .....and others

### **Impressum:**

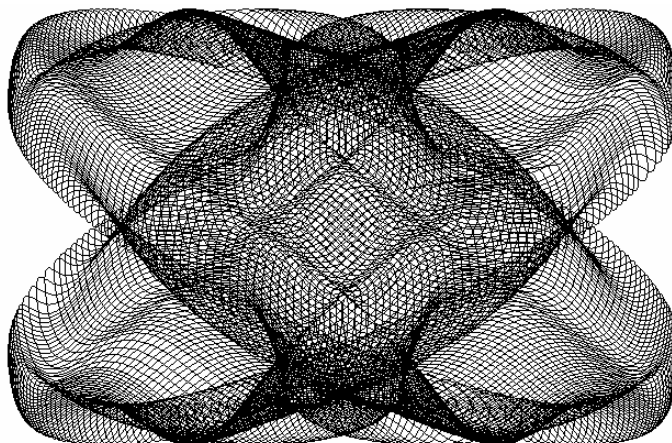
Medieninhaber: DERIVE User Group, A-3042 Würmla, D'Lust 1, AUSTRIA  
Richtung: Fachzeitschrift  
Herausgeber: Mag. Josef Böhm  
Herstellung: Selbstverlag

Alfonso J. Población, Valladolid, Spain

I send you a parametric plot with DERIVE. It seems to be a tridimensional net that was found by chance by Stanley S. Miller, Concord, Massachusetts. (I found it in Scientific American). Its equations are:

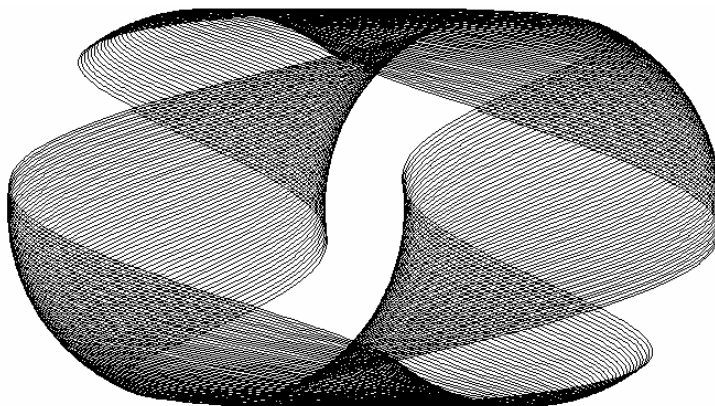
$$[\sin(0.99t) - 0.7 \cos(3.01t), \cos(1.01t) + 0.1 \sin(15.03t)]$$

It is not known if it is closed, cyclic or infinite. This plot is made with  $t \in [-300, 300]$ , but you can increase these values without finishing. (I tried up to  $[-1000, 1000]$ ). Making little variations in the coefficients, we can find some other fascinating and rare curves. I hope you will enjoy it.



$$\#1: [\sin(0.99 \cdot t) - 0.7 \cdot \cos(3.01 \cdot t), \cos(1.01 \cdot t) + 0.1 \cdot \sin(15.03 \cdot t)]$$

$$\#2: [\sin((1 - a) \cdot t) - b \cdot \cos(3.01 \cdot t), \cos((1 + a) \cdot t) + c \cdot \sin(15.03 \cdot t)]$$



$$a = c = 0, b = 0.8$$

Glyn D Williams, Gwynedd, Wales

I recently upgraded to **DERIVE 3.0XM**, to make better use of the extended memory on my computer and to take advantage of the extra facilities on **DERIVE 3.0**. In the pre-release notes, a copy of which was given to me at the conference at Plymouth, the facility of changing the menu to suit one's own needs is mentioned. Unfortunately the manual does not explain how to do this, and I would like to do this because I feel that the default menu is in parts badly thought-out, with commands like **Save** which are likely to be needed frequently on a lower menu, and **approX** having an awkward letter X. I know that the menu is kept in a file called **DERIVE.MEN**, but my attempts to adjust the menu have usually led to the computer locking up and needing to be rebooted. I should be grateful for any information on this, because I am finding it very frustrating that I cannot adjust the menu to suit my own way of working (and my left-handedness!)

**DNL:** I have sent Mr Williams the information he wants and I announce for the next DNL to publish the way to "tune" one's own menu.

**Dr. Klaus Kuenzer, Bruneck, Italy**

I've read Eugenio Roanes' contribution about the *Operations on Polynomials* in DNL#15 very keen. I enjoyed it very much, because he used recursive algorithms. I tried the functions and had a lot of troubles with calculation times, so it needed for MCD(32456,28588) 28.8 sec on a 486DX, or MCD\_POL(....) 1021.3 sec instead of the 4.8 sec in DNL#15. So I tried another way.  
(Dr. Kuenzer explained his ideas by one example; he simulates a kind of WHILE - DO loop. See here his functions. I checked Eugenio's times and can confirm them. ed.)

$$\text{rest\_}(a, b) := a - b \cdot \text{FLOOR}\left(\frac{a}{b}\right)$$

$$\text{ggt}(a, b) := (\text{ITERATE}(\text{IF}(x_2 = 0, x_1, \left[\frac{x_1}{x_2}, \text{rest\_}(x_1, x_2)\right]), x_1, [a, b]))_1$$

$$\text{ggt}(11^{100}, 13^{100}) = 1$$

2.9 sec in 1994, 0.05 sec in 2008

$$\text{infop}(p, v) := \text{ITERATE}\left(\text{IF}\left(u_1 = 0, u_1, \left[\frac{u_1 - \lim_{v \rightarrow 0} u_1}{v}, 1 + \frac{u_1}{2}, \frac{u_1}{1}\right], \left[\frac{u_1 - \lim_{v \rightarrow 0} u_1}{v}, 1 + \frac{u_1}{2}, \frac{u_1}{1}\right]\right), u_1, [p, -1, 0]\right)$$

$$\text{pgrad}(p, v) := (\text{infop}(p, v))_2$$

$$\text{pgrad}\left((3 \cdot x^3 + 2 \cdot x - 1) \cdot \left(3 \cdot x - 3 \cdot x^3 + 2 \cdot x^4 - \frac{x^5}{2}\right), x\right) = 8$$

0.7 sec in 1994, 0.02 sec in 2008

$$\text{lkoeff}(p, v) := (\text{infop}(p, v))_3$$

$$\text{lkoeff}\left((3 \cdot x^3 + 2 \cdot x - 1) \cdot \left(3 \cdot x - 3 \cdot x^3 + 2 \cdot x^4 - \frac{x^5}{2}\right), x\right) = -\frac{3}{2}$$

0.9 sec in 1994, 0.03 sec in 2008

$$\text{rest\_pol}(p, q, v) := \left(\text{ITERATE}\left(\text{IF}\left(\text{pgrad}(u_1, v) < \text{pgrad}(u_2, v), u_1, \left[\frac{u_1 - \frac{u_2 \cdot \text{lkoeff}(u_1, v)}{\text{lkoeff}(u_2, v)} \cdot \text{pgrad}(u_1, v) - \text{pgrad}(u_2, v)}{v}, u_2\right]\right), u_1, [p, q]\right)\right)_1$$

$$\text{rest\_pol}(8 \cdot x^3 - 26 \cdot x^2 - 15 \cdot x + 63, 12 \cdot x^3 - 49 \cdot x^2 + 25 \cdot x + 42, x)$$

$$\frac{5 \cdot (4 \cdot x^2 - 19 \cdot x + 21)}{3}$$

1.3 sec in 1994, 0.03 sec in 2008

$$\text{quo\_pol}(p, q, v) := \frac{p - \text{rest\_pol}(p, q, v)}{q}$$

$$\text{quo\_pol}(8 \cdot x^3 - 26 \cdot x^2 - 15 \cdot x + 63, 12 \cdot x^3 - 49 \cdot x^2 + 25 \cdot x + 42, x) \\ \frac{2}{3}$$

$$\text{ruffini}(p, a, v) := \left[ \frac{p - \lim_{v \rightarrow a} p}{v - a}, \lim_{v \rightarrow a} p \right]$$

$$\text{ruffini}(3 \cdot x^7 - x^5 + 2 \cdot x^4 - x^2, 3, x)$$

$$\left[ 3 \cdot x^6 + 9 \cdot x^5 + 26 \cdot x^4 + 80 \cdot x^3 + 240 \cdot x^2 + 719 \cdot x + 2157, 6471 \right]$$

$$\text{ggt\_pol}(p, q, v) := (\text{ITERATE}(\text{IF}(u_2 = 0, u, \left[ u_2, \text{rest\_pol}(u_1, u_2, v) \right]), \left[ u_2,$$

$$\text{rest\_pol}(u_1, u_2, v) \right], u, [p, q]))$$

$$\text{ggt\_pol}(p, q, v) := (\text{ITERATE}(\text{IF}(u_2 = 0, u, \left[ u_2, \text{rest\_pol}(u_1, u_2, v) \right]), \left[ u_2,$$

$$\text{rest\_pol}(u_1, u_2, v) \right], u, [p, q]))$$

$$\text{ggt\_pol}(x^4 - 7 \cdot x^3 + 18 \cdot x^2 - 22 \cdot x + 12, 3 \cdot x^3 - 13 \cdot x^2 + 8 \cdot x + 12, x)$$

$$\frac{34 \cdot (x^2 - 5 \cdot x + 6)}{9}$$

9.4 sec in 1994, 0.14 sec in 2008

### Al Rich, SWH, Hawaii

.... In order to get the background color saved in an INI file to be used by DERIVE, the INI file must be loaded along with DERIVE when it begins execution. In Version 3 this can be done by following DERIVE on the DOS command line with the name of the INI file. For example the command

DERIVE REVERSE.INI ARITH.DMO

starts DERIVE, initializes DERIVE using the settings in REVERSE.INI, and then runs the ARITH.DMO demonstration file. (You might want to put a note to this effect in the DNL since it is not yet in the DERIVE User Manual). Aloha.

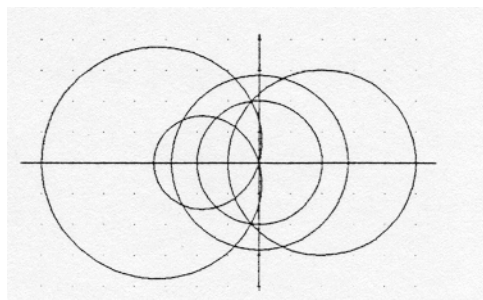
**DNL:** Thank you for the answer to the question how to save the background and work color of the plot screen using Transfer Save State and then calling these settings.

### Dr. H.-J. Kayser, Düsseldorf, Germany

Dr. Kayser wrote a short comment on Terence Etchell's INV NORM from DNL#15. In the next DNL I will publish his *An alternative to the function INV NORM from DNL#15*.

**Peter Baum, Kassel, Germany**

Peter Baum dealt with Th. Weths "Lexicon of Curves" and sent an extensive completion together with a couple of TIF files of nice pictures. Mr Baum uses only the polar form and he yields interesting results. The paper is too large to be part of the User Forum, so it will be a contribution in 1995. But I would like to include one of his plots:

**Message 3225: From SOFT WAREHOUSE to PUBLIC about INSTALLING DERIVE ON A LAN**

The compact DERIVE and DERIVE XM executable files are extremely small by today's standards. Also neither program uses overlay files while running (i.e. once loaded, they make no further access to the hard or floppy disk). Both these facts make installing and running DERIVE and DERIVE XM on a Local Area Network (LAN) simple and straight-forward.

The following describes how to install DERIVE on a LAN. The procedure for installing DERIVE XM is the same, except where noted:

1. Login with system supervisor privileges and create a directory named DERIVE on the LAN file server. If installing DERIVE XM, name the directory DERIVEXM to avoid a possible name conflict with an existing DERIVE directory. Give DERIVE users "read only" rights to this directory and give the system supervisor all rights (see your LAN manual for details). Note that it is NOT necessary to establish a command file search path to this directory.
2. Using the MS-DOS COPY or XCOPY commands, transfer all the files on the DERIVE or DERIVE XM distribution diskette to the new directory.
3. DERIVE users often need to save mathematical expressions and/or graphics screen images in files. Normally these files should NOT be stored in the DERIVE directory on the file server. Instead they should be saved in a working directory on a diskette, in a directory on the user's own computer, or in the user's own directory on the file server. The DOS drive and directory current at the time DERIVE is started is the default working directory. Therefore, DERIVE should be started from the desired working directory and NOT from the DERIVE directory on the file server.
4. On a LAN, it is usually preferable to start DERIVE using a DOS batch file. In addition to starting DERIVE, the batch file can automatically switch to the desired working directory and/or redirect printer output to the desired network printer. The batch file should be named DERIVE.BAT and saved on the file server in the standard batch file directory or any directory to which a command file search path has been established.

For example, if the file server is drive F, the batch file

```
@ECHO OFF
F:\DERIVE\DERIVE.EXE %1
```

starts DERIVE using the current DOS drive and directory as the working directory. If installing DERIVE XM, change the last line of the batch file to read

```
F:\DERIVEXM\DERIVE.EXE %1
```

Alternatively, the batch file

```
@ECHO OFF
C:
CD C:\DERIVE
F:\DERIVE\DERIVE.EXE %1
```

makes the DERIVE directory on drive C the working directory.

Finally, the batch file



@ECHO OFF

F:

CD F:\WORKING

F:\DERIVE\DERIVE.EXE %1

makes the WORKING directory on the file server the working directory.

5. If you want to redirect DERIVE printer output to a network printer, you can include the appropriate printer redirection command before the DERIVE.EXE line in DERIVE.BAT (see your LAN manuals for details). For example, if you are running a Novell LAN and the Netware program CAPTURE.EXE is in the PUBLIC directory on drive F, the command line

**F:\PUBLIC\CAPTURE L=1 Q=name NT NFF TI=5**

redirects DERIVE printer output from the LPT1 port to the network printer named name. Also tabs are not expanded, extra form feed characters are not sent at the end of print jobs, and data is sent to the printer after A 5 second time-out.

6. If the DERIVE user does not specify a drive and/or directory when attempting to load a utility file, DERIVE searches for the file in
  - a. the DOS drive and directory current when DERIVE was started;
  - b. the \DERIVE directory on the current drive; and finally
  - c. the directory from which DERIVE was invoked.

Thus, to provide access for all users to a utility file, copy the file to the DERIVE directory on the file server. Then the file will be found by search path c above.

7. The directories listed above are also searched to find the DERIVE.INI initialization file when DERIVE is started, and to find the DERIVE.HLP help file when a request is made for help (see the DERIVE User Manual for details).
8. You may want the state of DERIVE to be the same for all users when it starts. If this is other than the default initial state, save the desired state in a DERIVE.INI initialization file in the DERIVE directory on the file server (see Section 2.12 of the DERIVE User Manual for details). Then if individual users wish to modify the initial state, they can save their own DERIVE.INI files in their own working directory.

### **Message 3226: From SOFT WAREHOUSE to PUBLIC about COMPUTING LARGE POWERS**

Normally DERIVE raises numbers to integer powers. For example,  $2^3$  simplifies to 8. However, before computing powers, DERIVE determines if the answer is going to be too big to store in memory. For DERIVE, the largest power it will compute is roughly

$$2^{50000} = 16^{12500} \sim 3.16 \cdot 10^{15051}.$$

For DERIVE XM, the largest power it will compute is roughly

$$2^{400000} = 16^{100000} \sim 9.96 \cdot 10^{120411}.$$

Although it does not raise numbers to huge powers directly, DERIVE can be used to convert such expressions to scientific notation using common logarithms like we learned our high trig class.

For example, to convert  $3^{1000000}$ , set the precision level to 12 digits, set the notation to decimal, and approximate the expression

$$\text{LOG}(3^{1000000}, 10)$$

to give 477121.254719. Thus,  $3^{1000000}$  is approximately  $10^{0.254719} \cdot 10^{477121}$

Finally, highlight and approximate the first factor to give

$$1.7977 \cdot 10^{477121} \quad \text{which is } 3^{1000000} \text{ in scientific notation.}$$

Aloha, Al Rich, SWH.

|            |                                      |                 |
|------------|--------------------------------------|-----------------|
| <b>P 8</b> | <b>DERIVE Bulletin Board Service</b> | <b>D-N-L#16</b> |
|------------|--------------------------------------|-----------------|

**Message 3247: From SOFT WAREHOUSE to PUBLIC about 500,000 DIGITS OF PI**

In response to a challenge by a user, I had DERIVE XM compute  $\pi$  accurate to 500,000 digits on a 25mHz 486 with 4Mb of memory. It took about a week to compute and another week to save the result in a file.

It requires almost 50 pages to print using a very small type font. (No, I am not going to upload 500,000 digits on this BBS!) The last 20 of the 500,000 digits are

4042487602513819524

I would greatly appreciate someone independently (i.e. not using DERIVE XM) posting the last 40 of 500,000 digits of  $\pi$  on this BBS so I can confirm that the DERIVE result is correct. Please include your source.

Even on a 33mHz 486 with 8Mb of memory I was unable to get Maple V, Release 2 to compute more than about 100,000 digits of  $\pi$  before it exhausted memory. I would be interested to know if anyone can get Maple or Mathematica to compute 500,000 digits of  $\pi$ , and how much time and memory it takes.

Aloha,  
Al Rich, Soft Warehouse, Inc.

**Message 3250: From MICHAELWALSH to SOFT WAREHOUSE about #3247 / 500,000 DIGITS OF PI**

Hi Al--I've been involved in a discussion about computing  $\pi$  on Prodigy.

I know that  $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 \dots$  (this is a Taylor expansion of  $\arcsin(1)$ ) But this alternating series doesn't converge very rapidly. My question is: what algorithm did you use to compute 0.5M digits of  $\pi$ ? Thanks for your help. I've always been interested in number theory. Mike--Detroit

**Message 3255: From SOFT WAREHOUSE to MICHAELWALSH about #3250 / 500,000 DIGITS OF PI**

The algorithm DERIVE uses to compute  $\pi$  is based on Ramanujan's formula published in the Quarterly Journal of Pure and Applied Math, Volume 45, page 350, 1914. The formula is that  $4/\pi$  equals the sum of the terms

$$\frac{(-1)^m (1123+21460m) (1*3*5 \dots (2m-1)) (1*3*5 \dots (4m-1))}{882^{(2m+1)} 32^m (m!)^3}$$

as  $m$  goes from 0 to infinity. DERIVE then uses some "tricks" to compute the series efficiently.

I am still waiting for someone to post on this BBS the last 40 of 500,000 digits of  $\pi$  so I can verify DERIVE XM's result and sleep better at night.

Aloha,  
Al Rich, Soft Warehouse, Inc.

**Message 3256: From SOFT WAREHOUSE to PUBLIC about MYSTERIOUS TRIG IDENTITY**

In response to a user inquiry, I stumbled across the following trig identity

$$\begin{aligned} 2 \sin(\pi/14) \cos(\pi/7) &= \cos(\pi/7) - 1/2 && \text{or equivalently} \\ \sin(3\pi/14) - \sin(\pi/14) &= \cos(\pi/7) - 1/2 \end{aligned}$$

It looks like it should be easy to derive using the various rules for simplifying trig products, half-angles, etc. However, I have been stumped at finding a derivation.

If there is a generalization of this identity, it could be incorporated into DERIVE's trig simplifier. Then expressions on the LHS of the identity could be transformed to the obviously simpler equivalent expressions on the RHS.

However, the first step to finding such a generalization is to derive the above identity. I would greatly appreciate if someone could provide a derivation and/or generalization of this identity.

Aloha,  
Al Rich, SWH.

**Message 3263: From HADUD to SOFT WAREHOUSE about TRIG.IDENTITY**

Your formula can be derived from the trivial equality

$$\sin\left((7-3)\frac{\pi}{14}\right) = \cos\left(\frac{3\pi}{14}\right).$$

For brevity let  $E = e^{\frac{i\pi}{14}}$ . Then the LHS may be written  $\frac{E^4 - E^{-4}}{2i} = \frac{(E + E^{-1})(E^3 - E + E^{-1} - E^{-3})}{2i}$ .

$$\text{Thus } \sin\frac{3\pi}{14} - \sin\frac{\pi}{14} = \frac{\cos\frac{3\pi}{14}}{2\cos\frac{\pi}{14}} = 2\cos^2\frac{\pi}{14} - \frac{3}{2} = \cos\frac{2\pi}{14} - \frac{1}{2}. \quad \text{Q.E.D.}$$

I cannot believe that this is so clear for the average student. In my opinion it needs some knowledge, experience, skills and phantasy in manipulating trig expressions, Josef

What is necessary to know?

$$\sin x \cos y = \frac{\sin(x+y) - \sin(y-x)}{2} \quad (1)$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad (2)$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad (3)$$

$$(a^4 - b^4) = (a+b)(a^3 - a^2b + ab^2 - b^3) \quad (4)$$

$$\cos(2x) = 2\cos^2(x) - 1 \quad (5)$$

$$\cos(3x) = 4\cos^3 x - 3\cos x \quad (6)$$

$$2\sin\frac{\pi}{14}\cos\frac{2\pi}{14} = \sin\frac{3\pi}{14} - \sin\frac{\pi}{14} \quad (1)$$

$$\sin\left((7-3)\frac{\pi}{14}\right) = \sin\left(\frac{7\pi}{14} - \frac{3\pi}{14}\right) = \sin\left(\frac{\pi}{2} - \frac{3\pi}{14}\right) = \cos\left(\frac{3\pi}{14}\right) \quad (2)$$

$$\begin{aligned} \cos\left(\frac{3\pi}{14}\right) &= \sin\left(\frac{4\pi}{14}\right) = \frac{e^{\frac{4\pi i}{14}} - e^{-\frac{4\pi i}{14}}}{2i} = \frac{E^4 - E^{-4}}{2i} = \frac{(E + E^{-1})(E^3 - E^2 \cdot E^{-1} + E \cdot E^{-2} - E^{-3})}{2i} = \\ &= \left(\frac{E^3 - E^{-3}}{2i} - \frac{E - E^{-1}}{2i}\right)(E + E^{-1}) = \left(\frac{e^{\frac{3\pi i}{14}} - e^{-\frac{3\pi i}{14}}}{2i} - \frac{e^{\frac{\pi i}{14}} - e^{-\frac{\pi i}{14}}}{2i}\right) \cdot \left(e^{\frac{\pi i}{14}} + e^{-\frac{\pi i}{14}}\right) = \\ &= \left(\sin\left(\frac{3\pi}{14}\right) - \sin\left(\frac{\pi}{14}\right)\right) \cdot 2\cos\left(\frac{\pi}{14}\right) \end{aligned} \quad (3,4)$$

$$\begin{aligned} \left(\sin\left(\frac{3\pi}{14}\right) - \sin\left(\frac{\pi}{14}\right)\right) &= \frac{\cos\frac{3\pi}{14}}{2\cos\frac{\pi}{14}} = \frac{4\cos^3\frac{\pi}{14} - 3\cos\frac{\pi}{14}}{2\cos\frac{\pi}{14}} = 2\cos^2\frac{\pi}{14} - \frac{3}{2} = \\ &= 2\cos^2\frac{\pi}{14} - 1 - \frac{1}{2} = \cos\frac{2\pi}{14} - \frac{1}{2} = \cos\frac{\pi}{7} - \frac{1}{2} \quad \text{qued} \end{aligned} \quad (5,6)$$

We continue with Hadud's mail:

More generally, consider

$\sin\frac{(n-k)\pi}{2n} = \cos\frac{k\pi}{2n}$  with  $n, k$  both odd. Letting  $a = \frac{\pi}{2n}$  and proceeding as above we get

$$\sin((n-k-1)a) - \sin((n-k-3)a) + \dots \pm \sin a = \frac{\cos(ka)}{2\cos a}.$$

If  $k$  is fixed at 3 we have the identity

$$\sin((n-4)a) - \sin((n-6)a) + \sin((n-8)a) \cdots \pm \sin a = \cos(2a) - \frac{1}{2} \text{ with } a = \frac{\pi}{2n}.$$

By interchanging the roles of sines and cosines corresponding formulas may be found for cosine terms on the LHS.

**Message 3265: From HARALD LANG to SWH about #3256 / MYSTERIOUS TRIG IDENTITY**

Perhaps the following simple identity can help:

$$\sin\left(\frac{\pi}{14}\right) = \frac{\frac{1}{6} \cdot \cos\left(\frac{\text{ATAN}\left(\frac{3^{1/2} \cdot \left(-108 \cdot \cos\left(\frac{\pi}{7}\right)^2 + 108 \cdot \cos\left(\frac{\pi}{7}\right) + 5\right)^{1/2}}{9 \cdot \left(2 \cdot \cos\left(\frac{\pi}{7}\right) - 1\right)}\right) + \frac{\pi}{3}}{3}}{3} + \frac{\pi}{3}\right)$$

**Message 3266: From HARALD LANG to SWH #3256 / MYSTERIOUS TRIG IDENTITY**

More seriously, simplifying trig expressions enters when DERIVE solves 3- and 4-degree equations. For example, try solving

$$2z^3 - z^2 - z + \frac{1}{4} = 0.$$

In my version of DERIVE, I get three roots, one of which is (the other two look similar)

$$\text{APPROX}\left(\frac{\sqrt{7} \cdot \cos\left(\frac{\text{ATAN}\left(\frac{\sqrt{3}}{9}\right)}{3} + \frac{\pi}{6}\right)}{3} + \frac{1}{6}\right) = 0.9009688679$$

$$\text{APPROX}\left(\cos\left(\frac{\pi}{7}\right)\right) = 0.9009688679$$

But this is actually equal to  $\cos(\pi/7)$ , a much simpler expression. This follows easily from the SWH/Hans Dudler formula. (I just figured out that I should use "Std Char set" and print to a File in derive; then I can download the formula just as is!) -- Harald Lang

**DERIVE 6 produces another – but equivalent output for the solutions of the equation given above.**

$$\text{SOLUTIONS}\left(2 \cdot z^3 - z^2 - z + \frac{1}{4} = 0, z\right)$$

$$\left[\frac{\sqrt{7} \cdot \sin\left(\frac{\text{ATAN}\left(\frac{\sqrt{3}}{9}\right)}{3}\right)}{3} + \frac{1}{6}, \frac{1}{6} - \frac{\sqrt{7} \cdot \sin\left(\frac{\text{ATAN}\left(\frac{\sqrt{3}}{9}\right)}{3} + \frac{\pi}{3}\right)}{3}, \frac{\sqrt{7} \cdot \cos\left(\frac{\text{ACOT}\left(-\frac{\sqrt{3}}{9}\right)}{3}\right)}{3} + \frac{1}{6}\right]$$

[0.2225209339, -0.6234898018, 0.9009688679]

**Message 3269: From JERRY GLYNN to PUBLIC about 500000 DIGITS OF PI**

Al Rich recently calculated 500000 digits of pi on Derivexm. He asked for confirmation on this BBS. We are happy to announce his last 40 digits match up with two different sources so it look like congratulations are in order!!!

|                 |                                      |            |
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**Message 3284: From HARRY SMITH to SOFT WAREHOUSE about 500,000 DIGITS OF PI**

Al, I know your 500000 digits of  $\pi$  has been confirmed, but I thought you might be interested in my program for computing  $\pi$ .  $\pi$  computed accurate to 500,000 decimal digits has the following as its last 60 digits:

969593993754953622322221974659619332529074042487602513819524

The next 26 digits are: 2697391017563719753430045

This was computed by the program:

PiW - Compute PI to a million or so decimal places in  
Windows Version 1.00, last revised: 1992/12/16, 0600 hours,  
Copyright (c) 1981-1992 by author: Harry J. Smith,  
19628 Via Monte Dr., Saratoga, CA 95070.  
All rights reserved.

I used PiW to compute  $\pi$  to 500,000 decimal digits. This was done on an IBM AT compatible 33 MHz 486 DX computer using Windows 3.1 with 16 megabytes of RAM and 22 mega bytes of virtual memory. It took 37.3 hours for algorithm (a) and 31.0 hours for algorithm (b). The results were the same. The divide and square root routines have been improved since these runs, so the program is a little faster now (10 to 20%).

If I upload the program it will be in the two files PiW100.Zip and PiW500K.Zip.

Its online description is: PiW v1.00 By Harry J. Smith, Computes Pi to a million decimal places

Key words: Pi Math Windows C++ Precision FFT

PiW is a program to compute  $\pi$  to a million or so decimal places on an IBM compatible PC using Microsoft Windows for memory management. With 16 megabytes of RAM it can compute  $\pi$  to 524,200 decimal places. With 32 megabytes of RAM it can compute  $\pi$  to 1,048,500 decimal places.

PiW uses a Fast Fourier Transform (FFT) to speed up multiplications. It is written in Borland C++ for Windows. All source code is included. Also includes a copy of  $\pi$  to 500,000 places that was computed in 1.55 days using a 33 MHz i486. It is a good example of object oriented programming (OOP).

$\pi$  to 500,000 places is in file PiW500K.Zip. Every thing else is in file PiW100.Zip. -- Harry

**Message 3288: From CHRIS LAMOUSIN to JERRY GLYNN about PLOTTING**

I'm having trouble plotting a function with Derive. I'm still figuring the program out. The function is  $y=x - (3/2)x^{(2/3)}$ . I can get  $f(x)$  for  $x>0$ , but not for  $x<0$ . The program is set for all real numbers. Any Ideas?

**Message 3289: From JERRY GLYNN to CHRIS LAMOUSIN about #3288 / PLOTTING**

Yes I know what you need. Do Manage Branch Real and replot. Derive thinks correctly, that  $(-8)^{(2/3)}$  is a complex number. However this is an uncomfortable viewpoint in Calculus or before so Derive has this flexibility. You'll probably want to leave the setting this way so do Transfer Save State.

In commonly asked questions at the back of your manual you'll see this discussed.

**Message 3303: From JERRY GLYNN to PUBLIC about INTELLIGENCE?**

In all symbolic algebra programs I've tried  $2^{3001}/2^{3000}$  has produced a correct answer of 2 but has obviously done this by calculating both terms and dividing. We would not do it this way. All of these programs produce 2 as an answer to  $2^{(x+1)}/2^x$  so it's not that they can't do algebra but rather they can't do algebra when no variables are present. Could this ability pass for an example of human intelligence? Are there other examples out there? If we knew what humans were good at and what algebra systems were good at maybe we could see how to work together and then maybe train people for these jobs.

Do symbolic algebra systems simulate intelligence or are they just wonderfully fast calculators of symbolic expressions?

**Message 3304: From GREG SMITH to JERRY GLYNN about #3303 / INTELLIGENCE?**

The fact that you had to ask the question at all hints strongly at the answer ...

**Message 3308: From HARALD LANG to JERRY GLYNN about INTELLIGENCE?**

I have a similar experience. It seems like DERIVE has a difficulty to handle redundant information. I have pointed out earlier the curious fact that she correctly simplifies

$$(x + 2)^{1/3} + (8 \cdot x + 16)^{1/3} - (27 \cdot x + 54)^{1/3} = 0$$

to 0, but if you replace x by SQRT(5), then she fails! I don't know if this is typical for computer algebra programs in general, but no doubt we humans are not fooled by x being replaced by a number. -- Harald

**Message 3309: From JERRY GLYNN to HARALD LANG about #3308 / INTELLIGENCE?**

Excellent example ... again algebra is something with variable(s) and arithmetic is something without variables and they are treated differently.

When I wrote my comment I expected you to respond. How nice to be able to predict such a response at such a great distance. I'm inspired to think of more.

Any more responses out there?

Derive 6 works as expected:

$$(\sqrt{5} + 2)^{1/3} + (8 \cdot \sqrt{5} + 16)^{1/3} - (27 \cdot \sqrt{5} + 54)^{1/3} = 0$$

(In the next DNL you will find a fine example how an idea is growing on the Bulletin Board.)

## 2D Plots Labeling

Dr. Sergey V. Biryukov, Moscow

**Abstract**

Derive utility files for 2D plots labeling and axes drawing, numbering and labeling are described. Vector scaleable and rotatable fonts with ~250 characters are available. User fonts design is supported.

**Introduction**

Derive makes excellent 2D plots. It is for bright pupils, students and scientists. But ordinary pupils have some difficulties in keeping in mind plots and axes names, and in axes numbering calculation from scale values and cross position. Furthermore, axes cross is restricted to [0, 0] point. Our aim was to make 2D plots more clear and easy to percept.

**Utility overview**

Utility files set consists of: the main file `label.mth`, 5 font files `f_*.mth`, font designer `label_fd.mth` and appropriate document files `*.doc`. It supports plotting an arbitrary ASCII character (except graphic and control ones) or sequence of characters (text) in the 2D Plot window. Characters position, width, height, rotation (slant) and characters step in a text can be defined. Axes with arbitrary point of their cross, numbers near ticks and labels near arrows can be plotted.

**One character plotting**

`CHAR_(xy, wh, ch, rot)` is the base function of the utility. It prepares a set of points for character *ch* plotting in Options>Display>Points>Connected and Small mode. *xy* = [*x\_coordinate*, *y\_coordinate*] - lower-left point of the character rectangle, and *wh* = [*width*, *height*] defines rectangle width and height of this rectangle. Optional scalar parameter *rot* defines character rotation counterclockwise in radians. It is zero by default. It is necessary to preload a vector font (see next paragraph) to plot characters.

`CHAR_ ([1,2], [3,4], "A", 0)`, for example, simplifies to

$$[[4,2], [3.5,3.5], [2.5,6], [1.5,3.5], [1,2], [1.5,3.5], [3.5,3.5]]$$

and gives a 3 units width and 4 units height upper case letter "A". The lower-left corner of the letter is at the point [1,2].

### Vector Fonts

Characters in the utility are divided into several groups and combined in different combinations in 5 font files: Tiny (`f_t.mth`) - digits, arithmetic operations and letters x, y & z; Small (`f_s.mth`) - Tiny + English upper case letters; Small Russian (`f_sr.mth`) - Small + Russian upper case letters; Full (`f_f.mth`) - all characters except Russian letters; and Russian Full Font (`f_fr.mth`) - Full + upper & lower case Russian letters. All characters available (except Russian letters) are on Fig.1.

Each font file has two vectors: the vector of character names `chars_names` and the main vector `chars` of the form `[["character_name1", [point1, point2,...], ["char_name2",...],...]`. Points are in a standard for this font pattern rectangle. The size of this rectangle is defined in each font file. `[char_rect_width:=6, char_rect_height:=8]` for all our fonts. A lot of memory space was saved by defining character points as pairs of small integers but not rationals or decimals.

### Font designer

`Label_fd.mth` supports new font files design by: combining data from existing font files, adding/deleting new characters to/from font files, checking fonts & character points input in the form of long integer that reduces the number of keys to be pressed 3 times.

For example `N 60533813001353` simplifies to

`[6,0],[5,3],[3,8],[1,3],[0,0],[1,3],[5,3]]`

and gives an upper case letter "A" after plotting. Only nonnegative integer coordinates less than 10 can be input with `N()` function and the first point can't be `[0,0]`.

### Text Plotting (Labeling)

**LABEL(xy,wh,dxy,t,rot)** function prepares a vector of vectors of points for plotting text `t` given as vector of characters (`t = ["t", "e", "x", "t"]`),

`xy = [x,y]` - starting point,

`wh = [char_width, char_height]`,

`dxy = [char_step_x, char_step_y]`,

`rot` - characters rotation counterclockwise in radians (optional, default=0).

### Axes drawing, numbering and labeling

**DRAW\_AXES(xy1,xy2,axes\_cross,scale\_xy)**

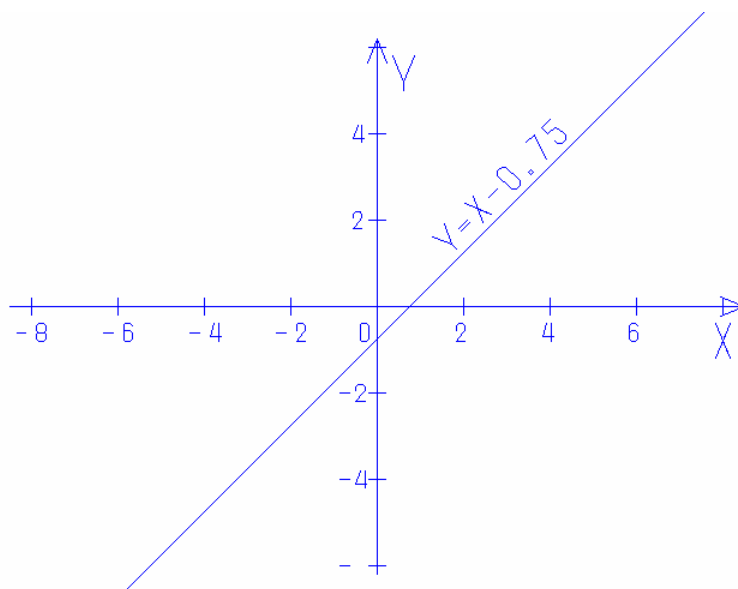
**= DRAW\_AXES([x1,y1],[x2,y2],[ac\_x,ac\_y],[sc\_x,sc\_y])**

prepares points for drawing axes (arrows points up & right) & axes ticks in the window with the lower - left corner coordinates `xy1 = [x1,y1]` & upper - right corner `xy2 = [x2,y2]`. `axes_cross` crosses at the point `[ac_x,ac_y]`. `scale_xy` defines distance between ticks on x & y axes. `ac_x/sc_x` & `ac_y/sc_y` must be integers.

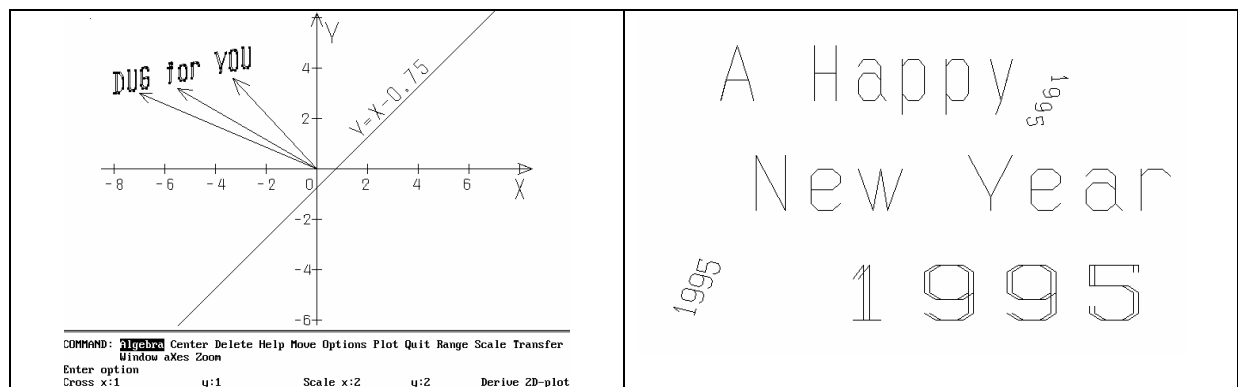
**LABEL\_AXES(xy2, axes\_cross, labels) = LABEL\_AXES[xy2, axes\_cross, [ wh\_x label\_x  
wh\_y label\_y ]]**

prepares points for labels near arrows. `wh_x` & `wh_y` are width & height of characters in x & y axes text labels `label_x` & `label_y`.

**NUMBER\_AXES(axes, numbers) = NUMBER\_AXES** 
$$\left[ \begin{bmatrix} xy1 \\ xy2 \\ axes\_cross \\ scale\_xy \end{bmatrix}, \begin{bmatrix} wh\_x & num\_x \\ wh\_y & num\_y \end{bmatrix} \right]$$





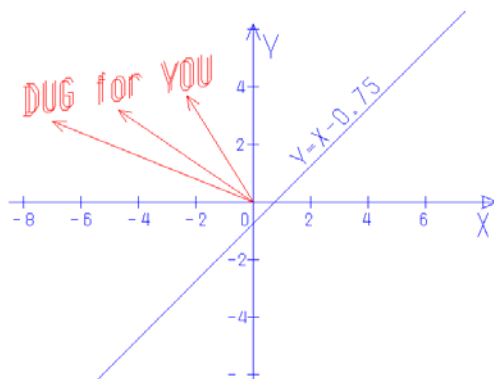


Screenshots from original DNL#16 (December 1994)

("DUG for YOU" and the arrows are added by me, then I tried to create a New Year's screen, Josef)

```
#8:  LOAD(D:\DfD\DNL\DNL94\MTH16\LABEL\F_E.MTH)
#9:  LABEL([-8, 3], [0.5, 1], [0.6, 0.1], [D, U, G, , f, o, r, , Y, O, U])
#10: SHIFT_LABEL(LABEL([-8, 3], [0.5, 1], [0.6, 0.1], [D, U, G, , f, o, r, , Y, O, U]), [0.1, 0.1])
#11: [ARROW([0, 0], [-7, 2.8], 0.5), ARROW([0, 0], [-4.7, 3.2], 0.5), ARROW([0, 0], [-2.3, 3.7], 0.5)]
#12: LOAD(D:\DfD\DNL\DNL94\MTH16\LABEL\F_E.MTH)
#13: LABEL([-3.5, 1.5], [0.6, 1], [0.8, 0], [A, , H, a, p, p, y])
#14: LABEL([-3, -0.5], [0.8, 1], [1, 0], [N, e, w, , Y, e, a, r])
#15: LOAD(D:\DfD\DNL\DNL94\MTH16\LABEL\F_NUMB3.MTH)
#16: LABEL([-1.5, -2.5], [1.2, 1], [1.5, 0], [2, 0, 0, 8])
#17: LABEL([-4, -2.5], [0.3, 0.4], [0.3, 0], [2, 0, 0, 8],  $\frac{\pi}{10}$ ,  $\frac{\pi}{3}$ )
#18: LABEL([2.5, 2], [0.2, 0.3], [0.3, 0], [2, 0, 0, 8],  $\frac{\pi}{5}$ ,  $-\frac{2\pi}{3}$ )
#19: SHIFT_LABEL(LABEL([-1.5, -2.5], [1.2, 1], [1.5, 0], [2, 0, 0, 8]), [0.05, 0])
#20: SHIFT_LABEL(LABEL([-1.5, -2.5], [1.2, 1], [1.5, 0], [2, 0, 0, 8]), [-0.05, 0])
#21: BOX([-5, -3.5], [5.5, 3.5])
```

*As you can see in the following figures, Sergey's tools are working with DERIVE 6 in the same way.*



|     |                             |          |
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### Some Additional info about 2D Plot Labeling Utility:

1. The Utility needs DERIVE 2.04 or later
2. Its main programming ideas:
  - FLOOR-function emulation. When using modern DERIVE versions, you have to exclude FLOOR & MOD-definitions and replace TRUNC by FLOOR & A\_V by APPEND.
  - Strings comparison and search in a vector.
  - Complex structured arguments, that simplifies using defined functions, and carries the idea of structuring not only functions but data also.
3. There are 3 additional useful functions in LABEL.MTH not described in the paper:
  - APPROW( $p1, p2, l$ ) draws an arrow from point  $p1$  to point  $p2$  with  $l$  = arrow sides length.
  - BOX( $p1, p2$ ) draws a rectangle with lower left corner  $p1$  and upper right corner  $p2$ .
  - SHIFT\_LABEL( $v, sxy$ ) shifts a vector of broken lines  $v$  (letters or curves) by  $sxy = [x, y]$ .
4. In LABEL\_FD.MTH the functions INSERT\_ELEMENT( $v, i, a$ ), REPLACE\_EL( $v, i, a$ ) and REPETITIONS( $a, v$ ) improves the VECTOR.MTH utility file. The first is a complementary function to DELETE\_ELEMENT( $v, i$ ). It inserts expression  $a$  in a vector  $v$  at position  $i$ . The 2<sup>nd</sup> replaces the  $i$ -th element of  $v$  with expression  $a$ . The 3<sup>rd</sup> returns the number of elements of vector  $v$  identical to expression  $a$ .

### Conclusion

Utility described can be easily used on every IBM PC compatible computer. Only 2 files are to be loaded as utilities: label.mth & one of the font files (\*.mth). Only 2 functions are needed by the user: LABEL() - for writing graph labels & ALL\_AXES() for axes drawing, labeling and numbering. Full Fonts are rather slow on IBM PC/XT, so, Tiny & Small Fonts for these computers are highly recommended.

### Acknowledgments

My thanks to Dr. B. Kutzler for the idea of writing this paper, V.B.Biryukov for draft fonts design & Prof. N.V.Soina for friendly advices and support.

### Comments of the editor:

I do neither list LABEL.MTH nor LABEL\_FD.MTH, you can download the files from the subdirectory <LABEL>. Included are many files with font sets. I am adding here the most important names which are packed in separated mth-files (which must be preloaded before using). I had to change some 1994-files because the DOS-characters are not compatible with the Unicode-characters of DERIVE 6 – and it might be too complicated to include them now. This is the reason that I copy some screens with the respective characters.

f\_e.mth: English Upper case and Lower case letters

f\_el.mth: English Lower case letters

f\_eu.mth: English Upper case letters

f\_gr.mth: ["α", "β", "Γ", "δ", "ε", "θ", "μ", "π", "σ", "τ", "φ", "Ω", "η", "Σ",  
"gamma\_low", "fi\_low", "omega\_low", "delta\_up"]

See first the font sets from 1994 and then the changed sets adapted for DERIVE 6.

```
f_mef.mth: ["∞", "φ", "ε", "∩", "≡", "±", "≥", "≤", "∫", "J", "÷", "≈"]
```

```
f_mf2.mth: ["°", "•", "·", "√", "n", "2", "□"]
```

```
f_mfr.mth: ["≥", "≤", "∫", "J", "÷", "≈", "°", "•", "·", "√", "n", "2", "□"]
```

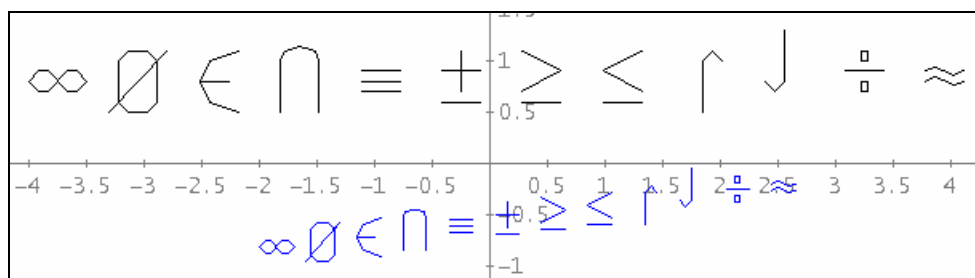
```
f_n0.mth: ["#", "$", "@", "[", "\", "]", "^", "'", "{", "|", "}", "~"]
```

```
#1: LOAD(D:\DFD\NLD\NLD94\MTH16\LABEL\LABEL.MTH)
```

```
#2: LOAD(D:\DFD\NLD\NLD94\MTH16\LABEL\F_MEF.MTH)
```

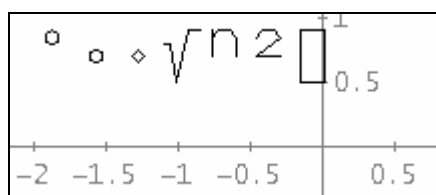
```
#3: LABEL([-4, 0.5], [0.5, 0.8], [0.7, 0], [∞, avg_, ε, ∩, id_, ±, ≥, ≤, int_u, int_l, div_, appr_])
```

```
#4: LABEL([-2, -1], [0.3, 0.5], [0.4, 0.05], [∞, avg_, ε, ∩, id_, ±, ≥, ≤, int_u, int_l, div_, appr_])
```



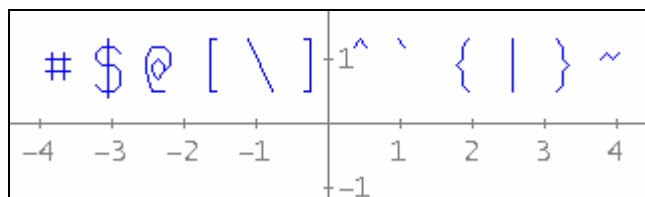
```
#5: LOAD(D:\DFD\NLD\NLD94\MTH16\LABEL\F_MF2.MTH)
```

```
#6: LABEL([-2, 0.5], [0.25, 0.4], [0.3, 0], [°, •, ·, √, n, 2, □])
```



```
#7: LOAD(D:\DFD\NLD\NLD94\MTH16\LABEL\F_N0.MTH)
```

```
#8: LABEL([-4, 0.5], [0.5, 0.8], [0.7, 0], [# , $ , @ , [ , \ , ] , ^ , ' , { , | , } , ~])
```



The next sets are still valid.

```
f_s.mth: ["!", "double_quotes", "%", "&", "'", "(", ")", "*", "+", ",", "-", ".", "/",
"0", "1", "2", "3", "4", "5", "6", "7", "8", "9", ":", ";", "<", "=", ">", "?", "_",
"A", "B", "C", "D", "E", "F", "G", "H", "I", "J", "K", "L", "M", "N", "O", "P", "Q",
"R", "S", "T", "U", "V", "W", "X", "Y", "Z"]
```

```
f_t.mth: ["!", "double_quotes", "%", "&", "'", "(", ")", "*", "+", ",", "-", ".", "/",
"0", "1", "2", "3", "4", "5", "6", "7", "8", "9", ":", ";", "<", "=", ">", "?",
" ", "X", "Y", "Z"]
```

It is important that the two lists (vectors) forming the font-set-mth-file are called **char\_names** (the short one) and **chars** (the huge one). Have a look at this vector and admire the enormous work lying in them. Maybe you will create another font set for us!! Editor.

## Der Fermat-Punkt im Dreieck The Fermat-Point in the Triangle

H.R.Geyer, Wiesbaden, GER

Während einer Projektwoche an unserer Schule (Gymnasium) im Frühjahr 1993 beschäftigte sich eine Gruppe von 7 Schülern der Jahrgänge 9 bis 13 mit den Einsatzmöglichkeiten von DERIVE im Mathematikunterricht. Bis auf einen Schüler hatten sie keine Erfahrungen mit dem Programm. Nach dem Einführungstag (4 Schulstunden) hatten sich alle soweit in die Bedienung eingearbeitet, dass sie in der Lage waren, sich an aktuellen Problemen des Mathematikunterrichts der jeweiligen Jahrgangsstufe zu versuchen.

Am 3. Tag stellte ich allen gemeinsam eine Aufgabe:

**Es ist die Lage desjenigen Punktes in einem (spitzwinkligen) Dreieck zu suchen, dessen Abstandssumme zu den drei Eckpunkten minimal ist - der Fermat - Punkt.**

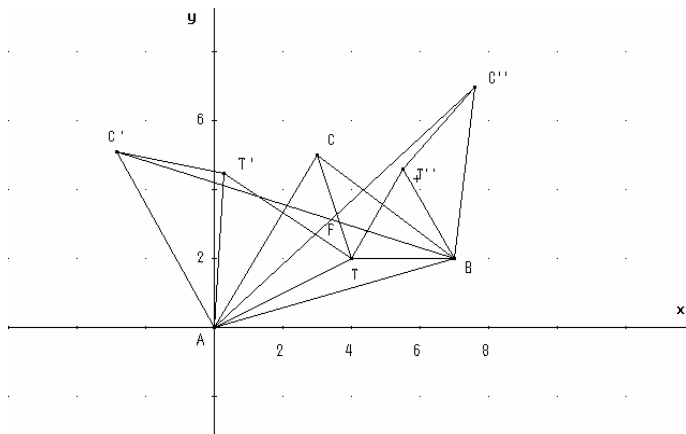
*During a project week at our school (grammar school) in spring 1993 a group of 7 students (age 15 - 19) dealt with DERIVE. After a short introduction - 4 hours - they were able to use DERIVE in connection with actual problems of their curriculum. On the 3rd day I set them all the same task:*

**Find the point in an acute triangle ABC with a minimal sum of distances to the points ABC - the Fermat point.** To explain the problem I showed the following proof by construction:)

Zur Erklärung führte ich den folgenden geometrischen Konstruktionsbeweis vor:

Sei T zunächst ein beliebiger Punkt im Dreieck  $\triangle ABC$ . Dreht man das Teildreieck  $\triangle CAT$  um  $60^\circ$  in positiver Drehrichtung um den Punkt A, so ergibt sich ein Polygonzug  $C'TTB$ , dessen Länge die gesuchte Abstandssumme ist. Die Länge des Polygonzuges ist minimal, wenn er auf der Geraden  $C'B$  liegt.

Analog dreht man das Teildreieck  $\triangle TBC$  um B um  $-60^\circ$ . Der gesuchte Fermat-Punkt F ergibt sich also als Schnittpunkt der beiden Geraden  $C'B$  und  $C''A$ .



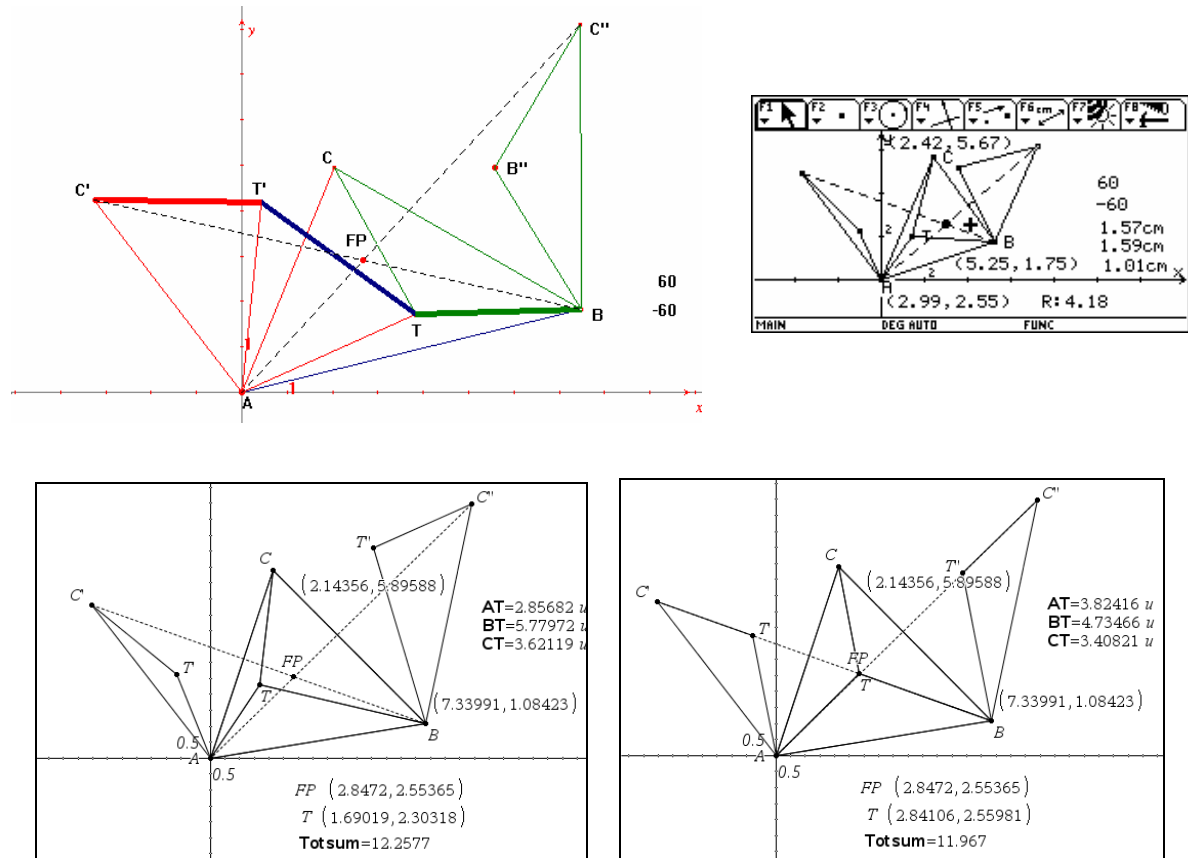
This is the original screen shot from 1994. The labels were produced using Sergey Biryukov's labeling tool.

*T is an arbitrary point in  $\triangle ABC$ . Having rotated  $\triangle CAT$  about  $60^\circ$  in positive direction you obtain the sum of the distances with the polygon  $C'TTB$ . Its length is minimal, if it coincides with the line  $C'B$ . In the same way you can rotate  $\triangle TBC$  round B by  $-60^\circ$  (negative). The Fermat - point is the intersection point of  $C'B$  with  $C''A$ .*

*It was interesting to observe how the pupils found different ways to solve the problem according to their mathematics knowledge.*

We can use dynamic geometry to illustrate the problem and to verify that intersection point FP is indeed the solution point:

I show the Cabri-figure (PC and TI-92/Voyage 200) and the TI-Nspire-figure(s), Josef.



Zur analytischen Bestimmung von F wurde vereinbart, das Dreieck in einem kartesischen Koordinatensystem mit A(0/0) zu beschreiben.

Die Schüler kamen nun selbständig je nach Kenntnisstand auf folgende Lösungsansätze:

### I. Berechnung des numerischen Abstandswertes als Funktion des Punktes F(x,y) und Minimieren durch sukzessive Einengung der Definitionsbereiche für x und y

*Calculation of the numerical value as a function of the point F(x,y) and minimization by a step by step restriction of the domains for x and y*

Definition of the TOTAL sum of distances:

#1: [b1:=, b2:=, c1:=, c2:=]

#2:  $da(x,y) := \sqrt{x^2 + y^2}$

#3:  $db(x,y,b1,b2) := \sqrt{(x-b1)^2 + (y-b2)^2}$

#4:  $dc(x,y,c1,c2) := \sqrt{(x-c1)^2 + (y-c2)^2}$

#5:  $TOTAL(x,y,b1,b2,c1,c2) := da(x,y) + db(x,y,b1,b2) + dc(x,y,c1,c2)$

|     |                                      |          |
|-----|--------------------------------------|----------|
| p20 | Heinz Rainer Geyer: Der Fermat-Punkt | D-N-L#16 |
|-----|--------------------------------------|----------|

Substitution of the co-ordinates of the given vertices of the triangle:

#6:  $L(x, y) := \text{TOTAL}(x, y, 7, 2, 3, 5)$

It follows an evaluation of function L in a 5 x 5 matrix,  
with increment h starting with x = s and y = t:

#7:  $\text{FERM}(s, t, h) := \text{VECTOR}(\text{VECTOR}(L(x, y), x, s, s + 4 \cdot h, h), y, t, t + 4 \cdot h, h)$

#8:  $\text{FERM}(3.41, 3.12, 0.002)$

#9: 
$$\begin{bmatrix} 10.30679813 & 10.30679185 & 10.30678803 & 10.30678668 & 10.3067878 \\ 10.30679055 & 10.30678457 & 10.30678106 & 10.30678002 & 10.30678146 \\ 10.3067845 & 10.30677882 & 10.30677562 & 10.3067749 & 10.30677665 \\ 10.30677998 & 10.30677461 & 10.30677173 & \underline{10.30677131} & 10.30677338 \\ 10.306777 & 10.30677194 & 10.30676936 & 10.30676927 & 10.30677165 \end{bmatrix}$$

#10:  $\text{FERM}(3.415, 3.127, 0.001)$

#11: 
$$\begin{bmatrix} 10.30676992 & 10.3067701 & 10.3067709 & 10.30677232 & 10.30677436 \\ 10.30676901 & 10.30676927 & 10.30677015 & 10.30677165 & 10.30677377 \\ 10.30676848 & 10.30676882 & 10.30676978 & 10.30677136 & 10.30677356 \\ \underline{10.30676834} & 10.30676876 & 10.3067698 & 10.30677146 & 10.30677373 \\ 10.30676858 & 10.30676908 & 10.3067702 & 10.30677194 & 10.3067743 \end{bmatrix}$$

Value of the distance-sum-function:

#12:  $L(3.415, 3.13) = 10.30676834$

Die beiden Beispiele aus dem DERIVE-Programm zeigen, daß die Auswertung der Matrizen zwar schwierig - weil unübersichtlich - ist, aber die Lage des Fermat-Punktes läßt sich so doch gut eingengen.

*Both examples show that the evaluation of the matrices is difficult because not easy to survey, but we can find the position of the Fermat point approximatively.*

The numerical approach would be easier to follow if we could include the x- and y-values as well in the table. This is possible with little manipulating the matrix – and the respective function could be provided as a tool by the teacher.

Producing a table including the x- and y-values:

#13:  $\text{values}(s, t, h) := \text{APPEND}([\text{APPEND}([], \text{VECTOR}(y, y, t, t+4h, h))], \text{FERM}(s, t, h))'$

#14:  $\text{values}(3.41, 3.12, 0.002)$

#15: 
$$\begin{bmatrix} & 3.41 & 3.412 & 3.414 & 3.416 & 3.418 \\ 3.12 & 10.30679813 & 10.30679185 & 10.30678803 & 10.30678668 & 10.3067878 \\ 3.122 & 10.30679055 & 10.30678457 & 10.30678106 & 10.30678002 & 10.30678146 \\ 3.124 & 10.3067845 & 10.30677882 & 10.30677562 & 10.3067749 & 10.30678146 \\ 3.126 & 10.30677998 & 10.30677461 & 10.30677173 & 10.30677131 & 10.30677338 \\ 3.128 & 10.306777 & 10.30677194 & 10.30676936 & 10.30676927 & 10.30677165 \end{bmatrix}$$

Then proceed as shown above.

## II. Differentiation der Abstandssummenfunktion

### (Differentiation of the distance-sum-function)

Für die Schüler der Klasse 11 lag es nahe, den Extrempunkt der Abstandssummenfunktion  $L(x,y)$  über die Ableitung zu suchen. Als Funktion zweier Veränderlicher entzieht sie sich allerdings dem für Schüler bekannten Lösungsverfahren. Einer der Schüler hatte hier die Idee, die partiellen Ableitungen von  $L(x,y)$  als Funktion der einen Variablen bei konstantem Wert der zweiten Variablen zu plotten. Hierzu definierte er einen Vektor, dessen Komponenten jeweils die so bestimmten partiellen Ableitungen waren. Die dargestellten Graphen führen zwar nicht direkt zu den Koordinaten des gesuchten Punktes, zeigen aber „auffälliges“ Verhalten in der Nähe der Koordinaten des Fermat-Punktes und insbesondere an den Grenzen der durch die Dreieckspunkte bestimmten Definitionsbereiche.

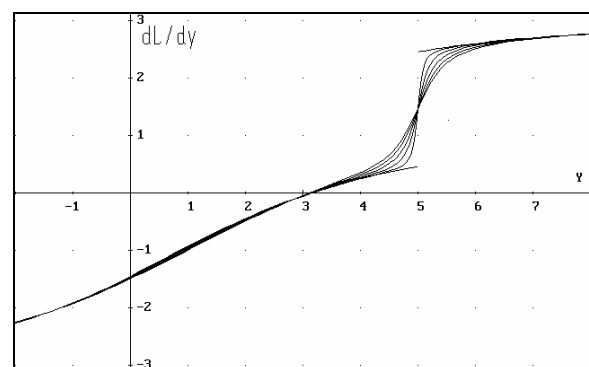
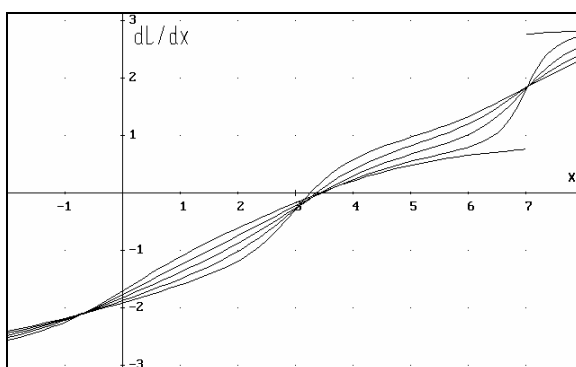
*The pupils of form 11 tried to find the extremal point of the distance sum function using the derivative. Being a function of two variables it exceeds the pupils' known algorithms. One of the students had the idea to plot the partial derivatives of  $L(x,y)$  wrt one variable keeping the other one constant. The graphs don't lead directly to the point we are looking for, but they are showing a very "strange" behaviour in the neighbourhood of the point and especially at the borders of the domain defined by the angles of the given triangle.*

Partial derivative of  $L(x,y)$  wrt  $x$ , which will be evaluated for  $2 \leq y \leq 4$ , with an increment of 0.5

$$\text{VECTOR}\left(\frac{d}{dx} L(x, y), y, 2, 4, 0.5\right)$$

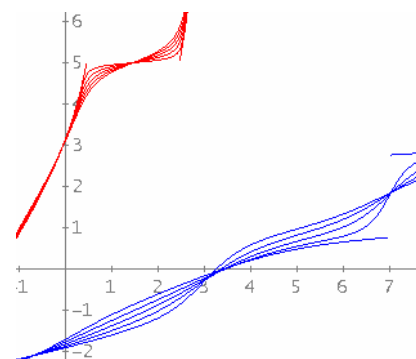
Partial derivative of  $L(x,y)$  wrt  $y$ , which will be evaluated for  $3 \leq x \leq 3.5$ , with an increment of 0.1

$$\text{VECTOR}\left(\frac{d}{dy} L(x, y), x, 3, 3.5, 0.1\right)$$



Original figures from DNL#16 (DOS-Derive)

Caution: The earlier Derive versions were not able to plot implicit functions, and they accepted all variables as the independent variable. Now we can do implicit plots and the graph of the 2<sup>nd</sup> family has another appearance:



## II. Nachbildung des geometrischen Beweises als Schnittpunktproblem zweier Geraden (Reproducing the geometric proof as intersection point of two lines)

Hierzu mußten die Koordinaten der gedrehten Punkte C' und C'' bestimmt werden. Die Schüler experimentierten dazu zunächst mit Tangens und Pythagoras. Zur Vereinfachung wurden dann die erforderlichen Drehungen nur auf Drehungen um den Ursprung (Punkt A) beschränkt. Außerdem stellte ich den Schülern die Matrix einer Drehung um den Ursprung und ihre Anwendung auf Ortsvektoren - sozusagen als "Black Box-Funktion" zur Verfügung. Entscheidend war, dass die Koordinaten der gedrehten Punkte C\_ und B\_ zur Verfügung standen.

Damit konnten die Geradengleichungen aufgestellt werden und das Schnittpunktproblem auf dem Niveau der Klasse 9 gelöst werden.

*The co-ordinates of the rotated points C' and C'' were necessary. At first the students experimented with Tangens and Pythagoras. As a simplification I made the rotation matrix available to the students - as a Black Box. Having the points C' and B' the problem was easy to solve for pupils even of level form 9.*

Δ ABC given by A(0,0), B(bx,by), C(cx,cy)

$$\#1: \text{line}(x1, y1, x2, y2, x) := \frac{y2 - y1}{x2 - x1} \cdot (x - x1) + y1$$

$$\#2: \text{triangle}(ax, ay, bx, by, cx, cy) := \begin{bmatrix} ax & ay \\ bx & by \\ cx & cy \\ ax & ay \end{bmatrix}$$

Rotation matrix around the origin = A(0,0)

$$\#3: \text{rot}(\alpha) := \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Calculation of rotated points C' and B' (c\_ and b\_):

$$\#4: c_ (cx, cy) := \text{rot}\left(\frac{\pi}{3}\right) \cdot [cx, cy]$$

$$\#5: c_ (cx, cy) := \left[ \frac{cx}{2} - \frac{\sqrt{3} \cdot cy}{2}, \frac{\sqrt{3} \cdot cx}{2} + \frac{cy}{2} \right]$$

$$\#6: b_ (bx, by) := \text{rot}\left(-\frac{\pi}{3}\right) \cdot [bx, by]$$

$$\#7: b_ (bx, by) := \left[ \frac{bx}{2} + \frac{\sqrt{3} \cdot by}{2}, \frac{by}{2} - \frac{\sqrt{3} \cdot bx}{2} \right]$$

Lines connecting B with C' (line G) and B' with C (line H):

$$\#8: G(x, bx, by, cx, cy) := \text{line}\left(\left(c_ (cx, cy)\right)_1, \left(c_ (cx, cy)\right)_2, bx, by\right)$$

$$\#9: H(x, bx, by, cx, cy) := \text{line}\left(\left(b_ (bx, by)\right)_1, \left(b_ (bx, by)\right)_2, cx, cy\right)$$

$$\#10: f\_pt(ax, ay, bx, by, cx, cy) := (\text{SOLUTIONS}(y = G(x, bx, by, cx, cy) \wedge y = H(x, bx, by, cx, cy), [x, y]))_1$$

Solving  $y=G$  and  $y=H$  for  $(x,y)$  gives the co-ordinates of the Fermat point.

Now let's try with our special values:



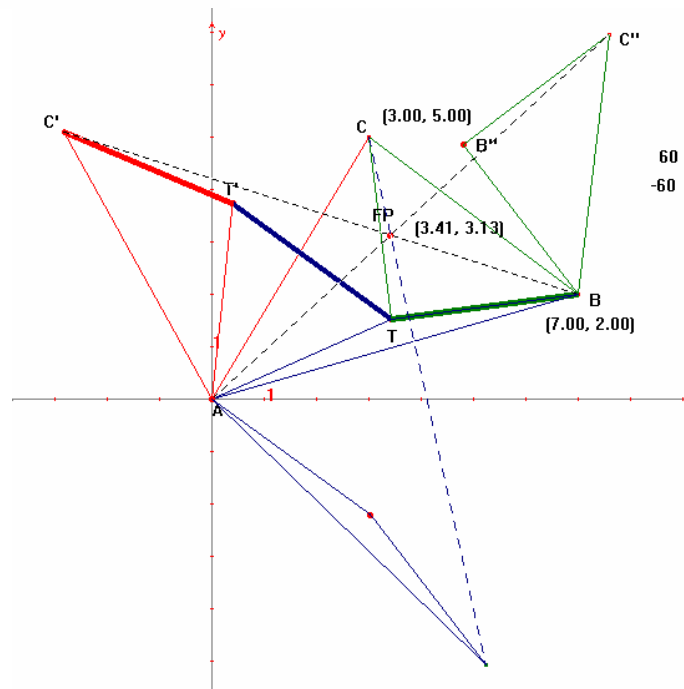
$$\#11: f_{pt}(0, 0, 7, 2, 3, 5) = \left[ \frac{5731}{1226} - \frac{2675 \cdot \sqrt{3}}{3678}, \frac{3229 \cdot \sqrt{3}}{3678} + \frac{1973}{1226} \right]$$

$$\#12: f_{pt}(0, 0, 7, 2, 3, 5) = [3.41483526, 3.129905399]$$

We repeat the calculation for the Nspire-Triangle from page 19:

$$\#14: [2.844703633, 2.550823696]$$

$$\#15: \text{triangle}(0, 0, 7, 2, 3, 5)$$



The Cabri-plot with the rotated triangle ATB (which is used for the proof)

Die Darstellungen zeigen deutlich, daß das Problem sicher nicht abschließend behandelt wurde. Es gibt noch viele Möglichkeiten zur Verfeinerung, zB beim Problem der Ableitungen oder bei der Schnittpunktsberechnung. Ich hoffe aber, daß deutlich wurde, wie ergiebig dieses Fermat-Problem für den Mathematikunterricht in vielen Klassenstufen ist und wie hervorragend sich dabei ein Programm wie DERIVE einsetzen läßt. Der experimentelle Charakter der mathematischen Arbeit in unserem Projekt bewirkte jedenfalls einen erkennbaren Motivationsschub, der dem Mathematikunterricht insgesamt bestimmt zu gute kommt.

*(The presentations are showing very clear, that we couldn't deal with the problem sufficiently. There are many possibilities for refinement. But I hope that I could show how productive this problem could be for many forms and how excellent you can use DERIVE for this purpose. The experimental character of the mathematical work in our project caused a significant motivation which could be very useful for maths teaching in total.)*

Additional comments for the revised version of DNL#16 will follow on the next page:

The students had the idea of applying calculus. So one could provide the suggestion to equate both partial derivatives to zero and ... then try to solve the resulting system of equations:

Inspecting the partial derivatives might discourage the students from solving the system. But they might also have heard or even learned to apply the Newton-Raphson-algorithm for approximatively solving equations. Why not extend for more variables??

Using the DERIVE-Help they might find `NEWTONS(u, x, x0)`.

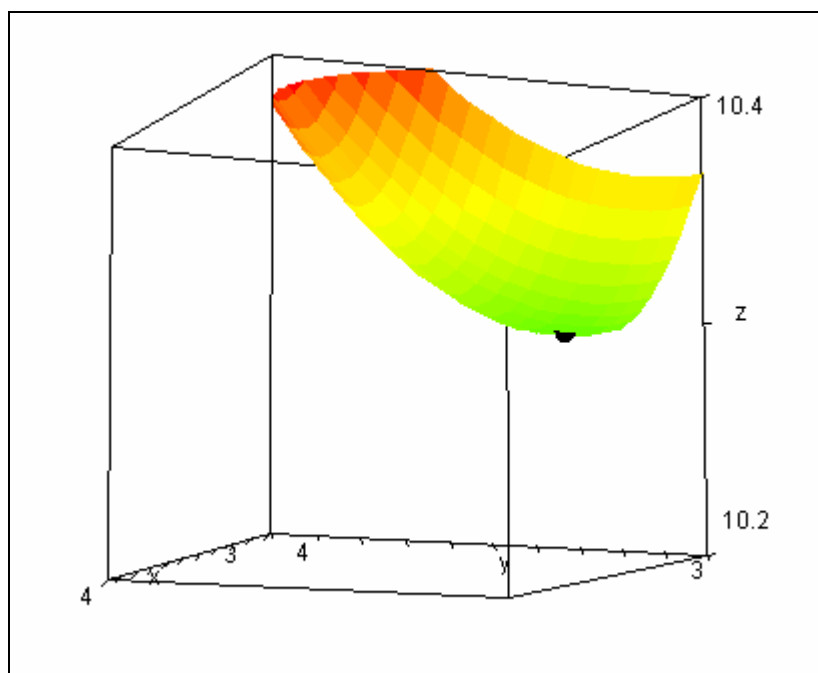
```
#32: PrecisionDigits := 6
#33:  $\left[ \text{REVERSE} \left( \text{NEWTONS} \left( \left[ \frac{d}{dx} L(x, y), \frac{d}{dy} L(x, y) \right], [x, y], [3, 3] \right) \right) \right]$ 
#34:  $\begin{bmatrix} 3.41483 & 3.12990 \\ 3.41483 & 3.12990 \\ 3.41483 & 3.12990 \end{bmatrix}$ 
```

Here we see the last three iterations of the algorithm.

And finally it might be a welcome surprise to show the solution in a 3D-environment.

$L(x,y)$  from expression #6 is a function of two variables which can be plotted immediately in the 3D-Plot window. The students have to find appropriate settings for the box.

Using the Trace tool they are asked to find the “Minimum Point” and then they can add the point  $(x_{FP}, y_{FP}, L(x_{FP}, y_{FP}))$ .



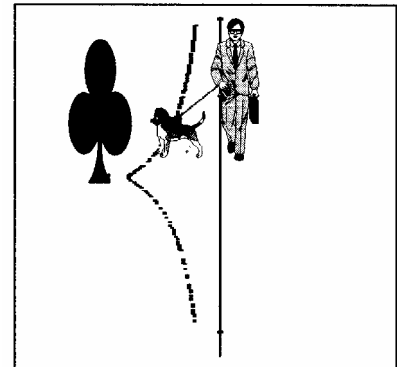
## Ebene Algebraische und Transzendente Kurven (5)

Thomas Weth, Würzburg, Germany

### Die Hundekurve - Die Konchoide des Nikomedes

Die in dieser Folge behandelten Kurven sind Erweiterungen der sog. „Hundekurve“ oder „Tractrices“. Tractrices entstehen, wenn ein Herrchen seinen Hund auf einem geraden Weg führt, während der Hund - bei konstanter Leinenlänge - seinem „Lieblingsbaum“ zustrebt. Mathematisch formuliert handelt es sich bei dem Weg des Hundes um eine spezielle Form von Konchoiden.

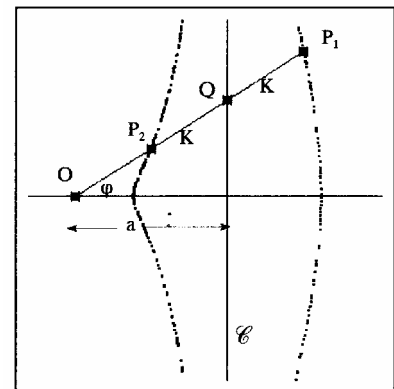
*The curves dealt with in in this sequel are extension of the so called "Dog's curve" or "Tractrices". They are created when a man walks along a line while his dog - kept on a lead with constant length - wants to visit his favourite tree. From the mathematical point of view the dog's way is a special form of a conchoid.)*



### Erzeugung von Konchoiden - *How to create a conchoid*

Konchoiden im allgemeinen ergeben sich nach folgender Konstruktionsvorschrift:

Gegeben ist eine ebene Kurve  $C$  und ein Punkt  $O$  in ihrer Ebene. Von einem Punkt  $Q$  der Kurve aus trägt man auf der Geraden  $OQ$  in die beiden möglichen Richtungen jeweils eine Strecke konstanter Länge  $K$  ab; die Endpunkte  $P_1$  und  $P_2$  dieser Strecken sind dann Konchoidenpunkte zur Kurve  $C$ .



*This is the instruction how to obtain a conchoid: given is a plane curve and a point  $O$  in its plane. Have any point  $Q \in C$ . The two points  $P_1$  and  $P_2$  on the line  $OQ$  with  $QP_1 = QP_2 = K$  (const) are points of a conchoid to  $C$ . If the curve  $C$  is a line the conchoids are especially simple, they are tractrices. Nikomedes (200 BC) invented this curve for doubling a die and for trisecting an angle. Its name is derived from the greek word for shell:  $\kappa\omicron\gamma\chi\eta$ .)*

### Die Konchoide des Nikomedes - Herleitung der algebraischen Kurvengleichung

Besonders einfache Konchoiden ergeben sich, wenn die Leitkurve  $C$  eine Gerade ist (wie im Beispiel der Tractrix). Diese Kurven wurden von Nikomedes (ca 200 v.Chr.) zur Würfelverdoppelung und zur Dreiteilung des Winkels erfunden. Der Name ist wegen der Kurvenform abgeleitet vom griechischen  $\kappa\omicron\gamma\chi\eta$ , Muschel.

Für die Kurvenpunkte  $P_{1/2}$  ergibt sich in Polarform (vgl. obige Zeichnung):

$P_{1,2} : r = \frac{a}{\cos \varphi} \pm K$  oder  $\left(r - \frac{a}{\cos \varphi}\right) \mp K = 0$  und daraus für die gesamte Kurve:

$$\left(r - \frac{a}{\cos \varphi} - K\right)\left(r - \frac{a}{\cos \varphi} + K\right) = 0 \quad \text{oder kurz} \quad \left(r - \frac{a}{\cos \varphi}\right)^2 = K^2.$$

(Polargleichung der Konchoide des Nikomedes; *this is the polar form of Nikomedes' conchoid. Now you easily can find the Cartesian form.:*)

Den Übergang zu kartesischen Koordinaten erhält man mit  $x = r \cos(\varphi)$  und  $y = r \sin(\varphi)$ :

$\left(r - \frac{a \cdot r}{x}\right)^2 = K^2$ . Ersetzt man nun noch  $r^2 = x^2 + y^2$  (DERIVE!) und faktorisiert die Summe, so erhält man die algebraische Kurvengleichung

$$(x^2 + y^2)(x - a)^2 = K^2 x^2.$$

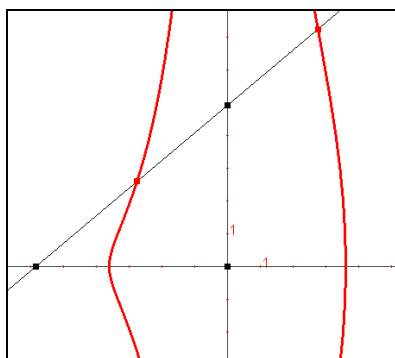
Aus der Kurvengleichung läßt sich nun entnehmen, dass die Kurven (für verschiedene  $K$ )

- symmetrisch zur x-Achse sind,
- die Leitgerade  $C$  mit der Gleichung  $x = a$  zur Asymptote haben und
- im Ursprung einen (evtl isolierter) Kurvenpunkten haben.

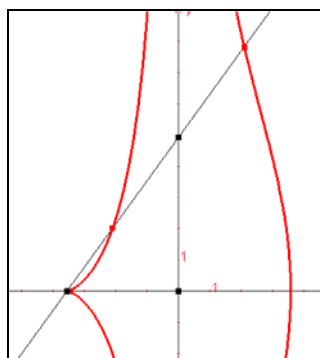
*(From the curve's equation you can see that for different values for  $K$ , the curves*

- *are symmetric in respect to the x-axis,*
- *have line  $C$  with  $x = a$  as an asymptote and*
- *the origin is a (eventually isolated) point of the curve.)*

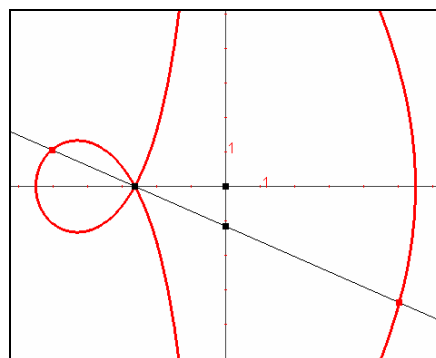
Für die Fälle  $K < a$ ,  $K = a$  und  $K > a$  ergeben sich die drei Kurvenformen:



$K < a$



$K = a$



$K > a$

*DERIVE 6 has two remarkable features which we can use to reproduce the construction of this curve: the slider bars and the possibility to switch between rectangle coordinates and polar coordinates in one session. This one more occasion to narrow the gap between computer algebra and dynamic geometry. I'll try to explain the procedure very briefly guided by the following screen shots, Josef*

Start in Rectangular coordinates.

Plot #1 and #2 and introduce slider bars for  $a$  ( $0 \leq a \leq 10$ ) and  $p_-$  ( $-\pi/2 \leq p_- \leq \pi/2$ ).

#1:  $x = a$

#2:  $x \cdot \tan(p_-)$

Moving the slider bars you can change  $a$  and see the rotating line.

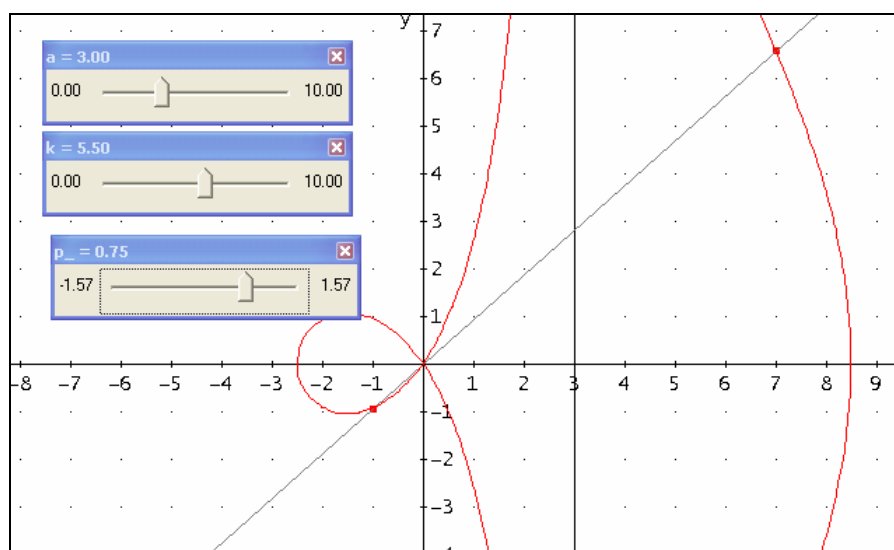
Change now to polar coordinates, introduce a third slider bar for  $k$  ( $0 \leq k \leq 10$ ).

Plotting #3 you can move the points on the two branches of the curve, exchanging  $p_-$  by  $\phi$  gives the polar form of the curve, and so does the implicit form in #4.

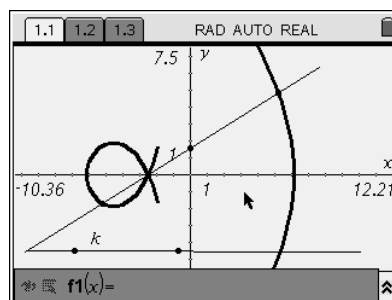
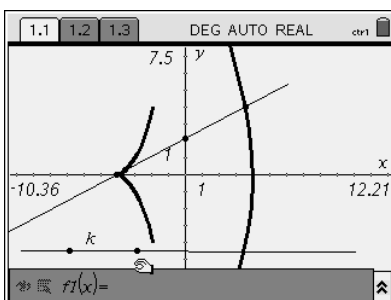
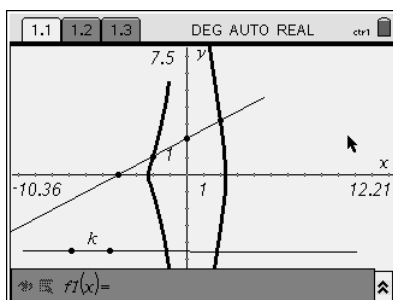
You can alter  $a$ ,  $k$  and  $p_-$  as you like.

#3: 
$$\begin{bmatrix} \frac{a}{\cos(p_-)} + k & p_- \\ \frac{a}{\cos(p_-)} - k & p_- \end{bmatrix}$$

#4: 
$$\left( r - \frac{a}{\cos(\phi)} \right)^2 = k^2$$



This is the “Dog’s curve” on the hand held TI-Nspire.



### Verwendung der Konchoide zur Würfelverdoppelung - *Doubling a die*

Zu konstruieren ist die Kantenlänge  $x$  eines Würfels, der den doppelten Inhalt besitzt wie ein Würfel mit der Kantenlänge  $a$ ; also  $x^3 = 2a^3$  oder  $x = a^3 \sqrt[3]{2}$ .

Dazu zeichnet man eine Konchoide mit  $K = a$  und zeichnet durch U unter  $60^\circ$  eine Gerade zur  $x$ -Achse, die den rechten Kurvenzweig in S schneidet. Die Gerade OS schneidet die Leitkurve C dann im Punkt T.  $x = OT$  ist dann die gesuchte Würfelkante mit  $x^3 = 2a^3$ .

$OT = x$  is the edge of the cube with  $x^3 = 2a^3$ .

Nach dem Strahlensatz ist nämlich

$$x : a = 2a : y \text{ bzw. } x \cdot y = 2a^2. \quad (I)$$

Mit dem Lot von O auf die Gerade US ist das Dreieck OSL rechtwinkelig und es gilt zunächst:

$$\overline{LU} = a \cos(60^\circ) = \frac{a}{2}.$$

Damit folgt für das Dreieck  $\triangle OSL$ :  $(x + a)^2 = \left(y + \frac{a}{2}\right)^2 + \overline{OL}^2$

und für das Dreieck  $\triangle OLU$ :  $a^2 = \frac{a^2}{4} + \overline{OL}^2$ .

Subtraktion liefert:  $(x + a)^2 - a^2 = \left(y + \frac{a}{2}\right)^2 - \frac{a^2}{4}$  bzw.  $x^2 + 2ax = y^2 + ay$

(I) und (II) vereinfacht man durch Elimination von  $y$  zu  $x^3 = 2a^3$ .

You can simplify (I) and (II) by eliminating  $y$  and you obtain  $x^3 = 2a^3$

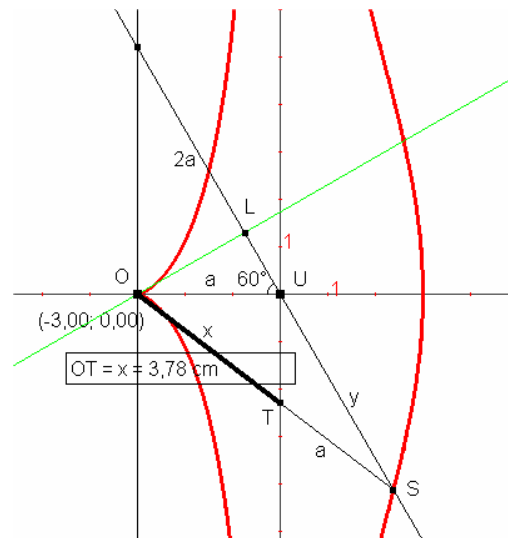
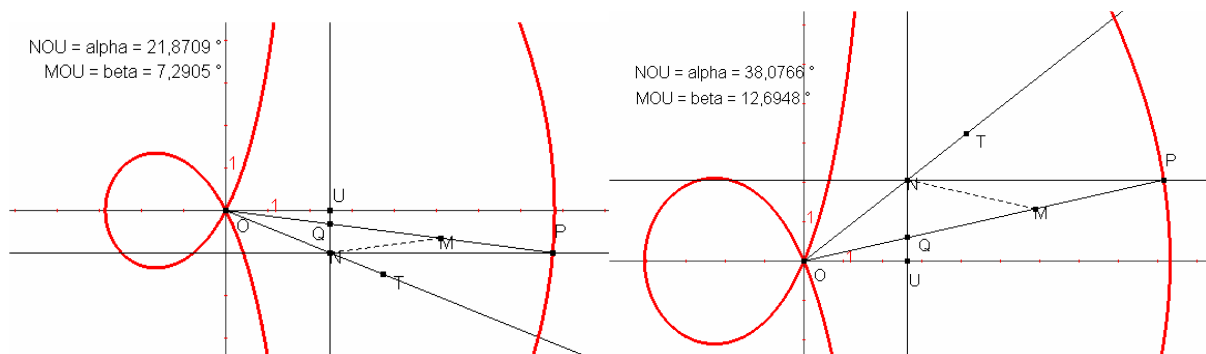
### Verwendung der Konchoide zur Drittelung des Winkels

#### How to use the conchoid to trisect an angle

Ist  $\alpha$  der zu drittelnde Winkel, so trägt man auf dem einen Schenkel eine beliebige Strecke [OU] mit der Länge  $a$  ab. Das Lot in U auf die Gerade OU schneidet den anderen Schenkel im Punkt N.

Mit  $ON = c$  zeichnet man zur Leitgeraden NU eine Konchoide mit O als Pol und  $K = 2c$ . Durch N

zeichnet man die Parallele zu OU, welche die Konchoide im Punkt P trifft. Dann ist:  $|\angle POU| = \frac{\alpha}{3}$ .



|          |                                      |     |
|----------|--------------------------------------|-----|
| D-N-L#16 | Thomas Weth: A Lexicon of Curves (5) | p29 |
|----------|--------------------------------------|-----|

*If you want to trisect  $\alpha$ , then take any arbitrary point  $U$  with  $OU = a$  on one side. The perpendicular intersects the other side of  $\alpha$  in  $N$ . With  $ON = c$  draw a conchoid with respect to the line  $NU$  with  $O$  as its pole and  $K = 2c$ . Let  $NP \parallel OU$ , then  $\angle POU$  is the third of  $\angle NOU$*

Begründung:

Ist  $M$  die Mitte von  $QP$ , so liegen  $P$ ,  $Q$  und  $N$  auf dem Thaleskreis über der Strecke  $[PQ]$ . Also ist  $PM = MQ = MN = c = ON$ . Damit gilt:

$$|\angle NPM| = |\angle MOU| = \beta \quad (\text{als „Z - Winkel“}).$$

Da das Dreieck  $MPN$  gleichschenkelig ist, gilt für den Außenwinkel  $\angle OMN$ :

$$|\angle OMN| = 2\beta.$$

Da außerdem das Dreieck  $OMN$  gleichschenkelig ist, gilt wegen der Kongruenz der Basiswinkel:

*( $M$  is in the middle of  $PQ$ , then  $P$ ,  $Q$  and  $N$  are lying on a Thales circle, so  $PM = MQ = MN = c = ON$ .*

*Then you will find two isosceles triangles  $\triangle MPN$  and  $\triangle OMN$ , giving simple relations between the angles.)*

$$|\angle MON| = |\angle OMN| = 2\beta. \text{ also ist } \alpha = 2\beta + \beta \text{ oder } \beta = \alpha/3.$$

Es sei hier ausdrücklich darauf hingewiesen, daß es sich bei den Konstruktionen zur Würfelverdopplung und zur Winkeldreiteilung nicht um elementare Zirkel- und Linealkonstruktionen im Platon'schen Sinne handelt. Wie man an den vorhergehenden Beispielen sieht, ist nämlich die Würfelverdopplung und die Winkeldreiteilung sehr wohl geometrisch möglich - nur eben nicht - wie zB in der Galois-Theorie bewiesen wird - mit Zirkel und Lineal alleine. In den obigen Konstruktionen wurde nämlich eine Gerade mit einer Kurve (der Konchoide) geschnitten, also eine Operation durchgeführt, die mit Zirkel und Lineal alleine nicht möglich ist.

Abschließend sei bemerkt, dass sich für einen Kreis als Leitlinie andere bekannte Konchoiden, u.a. die Pascalschen Schnecken und die Kardioide ergeben. Wir werden in einer späteren Folge darauf zu sprechen kommen.

*I want to emphasize that these constructions are not elementary constructions using ruler and compasses only in Platon's sense, because you need the intersection point between a conchoid and a line and this operation can't be done with the tools mentioned. The Galois Theory proofs this fact.*

*Finally I would like to mention, that using a circle as leading line we will find other well known conchoids as Pascal's snail and the cardioid. In a later sequel we will talk about.*

## DERIVE and plotting T - periodic functions

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### Introduction.

For most simple periodic functions it is rather easy to plot them. Think of  $y = \sin(x)$  for instance. Just Author  $\sin(x)$ , press twice on **Plot** and voila.

It is more difficult to plot functions like those in figure 1.

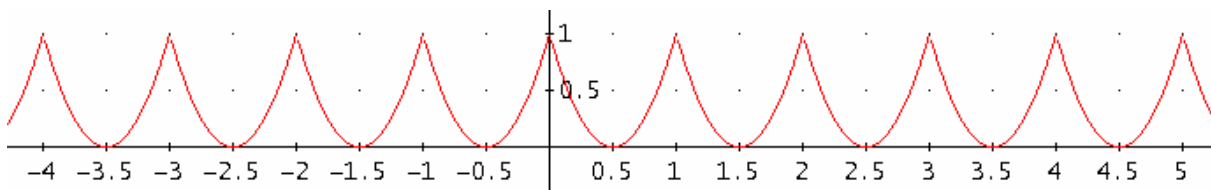


Figure 1: “difficult” periodic function.

We will discuss the possibilities and problems connected with plotting this kind of functions.

### The tools.

Derive knows the following functions:

- a)  $\text{CHI}(a, x, b)$ ; a function which returns 1 for  $x \in [a, b]$  and 0 outside  $[a, b]$
- b) The  $\text{IF}$  - function. We can use this function for piecewise defined expressions using conditions.  
For instance:  $\text{IF}(x < 0, x^2, x)$  is a function which for  $x < 0$  returns a parabola and for  $x \geq 0$  the straight line  $y = x$ .
- c)  $\text{MOD}(a, b)$  which approximates to **a** modulo **b**. When  $\text{MOD}(x, 1)$  is plotted, you get a saw-shaped-function.  $\text{MOD}(x, \pi)$  also returns a saw shaped function but now with period  $\pi$ .
- d)  $\text{FLOOR}(x)$  is narrowly connected to  $\text{MOD}(x)$ . Simplifying  $\text{MOD}(x, 1)$  yields  $x - \text{FLOOR}(x)$ .  
In other words:  $\text{MOD}(x, 1) = x - \text{FLOOR}(x)$ .

### Plotting using CHI, IF and MOD

#### The usage of CHI.

Let's look at the following piecewise defined function:

$$f(x) = x^2 \text{ if } x \in [0, 1] \text{ and } 0 \text{ if } x \notin [0, 1]$$

We could Author this with:  $f(x) := \text{CHI}(0, x, 1) \cdot x^2$ .

A short explanation:

The fact that the product is forced to 0 outside  $[0, 1]$  makes it rather difficult for us to plot functions only defined on an interval: DERIVE plots 0 outside the interval while the *function* is not defined over there.



Let's suppose we want a function like the one in figure 1. We can define  $f(x)$  on  $[0,1]$ , on  $[1,2]$  etc. The next step is to define this function in terms of **SUM**: Apparently it's not a problem when the number of terms is finite.

On  $[0,1]$  we get  $4 \text{ CHI}(0, x, 1) (x - 0.5)^2$ .

The quadratic function defined on  $[1,2]$  would be  $f(x) := 4 \text{ CHI}(1, x, 2) (x - 1.5)^2$ ;

the one on  $[k, k+1]$ :  $4 \text{ CHI}(k, x, k+1) (x - 0.5 - k)^2$ .

So when we take the first 5 terms we get:

$$f(x) := \sum_{k=0}^5 4 \text{ CHI}(k, x, k+1) (x - 0.5 - k)^2$$

This function isn't a true periodic function. To produce a true periodic function  $f(x)$  must become symmetric and extended to infinity (and negative infinity). So let us define  $g(x)$ :

$$g(x) := \sum_{k=-\infty}^{\infty} 4 \text{ CHI}(k, x, k+1) (x - 0.5 - k)^2$$

Although DERIVE is capable of plotting  $f(x)$ , it accepts  $g(x)$  but seems not capable of plotting it. Probably DERIVE tries to evaluate the function and can't handle infinity.

## Using IF

What about the IF - function? We can use this function to make a piecewise defined function: for instance:  $f(x) := \text{IF}(x < 0 \text{ OR } x > 1, x, x^2)$ .

A short explanation:

When  $x < 0$  or  $x > 1$ , then  $f(x) = x$ , everywhere else  $f(x) = x^2$ .

Another example:  $g(x) := \text{IF}(x < -1 \text{ AND } x < 2, x, ?)$

When we plot  $g(x)$  we can see that outside  $[-1,2]$   $g(x)$  is not forced to 0 because the function is not defined there.

Nevertheless we have the same problems with the "IF - command" as those we encountered with the CHI-functions: it is not possible to plot true piecewise defined functions although it is possible to define them!

## Using MOD(a,b)

Because  $\text{MOD}(x, a)$  already is a periodic function, it offers more perspectives. Moreover, it is not necessary for  $a$  to be an integer: for instance  $\text{MOD}(5, 2.1)$  will yield 0.8.

For students this function often gives surprises. It's not automatically clear for them why the plot of  $x \text{ MOD}(x)$  or  $\text{MOD}(x^2)$  looks so strange. These plots invite the students to discuss simplifications in DERIVE involving  $\text{MOD}(x)$ . Can we understand what is happening here? What is the connection between  $\text{FLOOR}(x)$  and  $\text{MOD}(x)$ ? After some experimenting in DERIVE they find it rather easy to understand that  $\text{MOD}(x) = x - \text{FLOOR}(x)$  and  $x \text{ MOD}(x)$  simplifies to  $x^2 - x \text{ FLOOR}(x)$ .

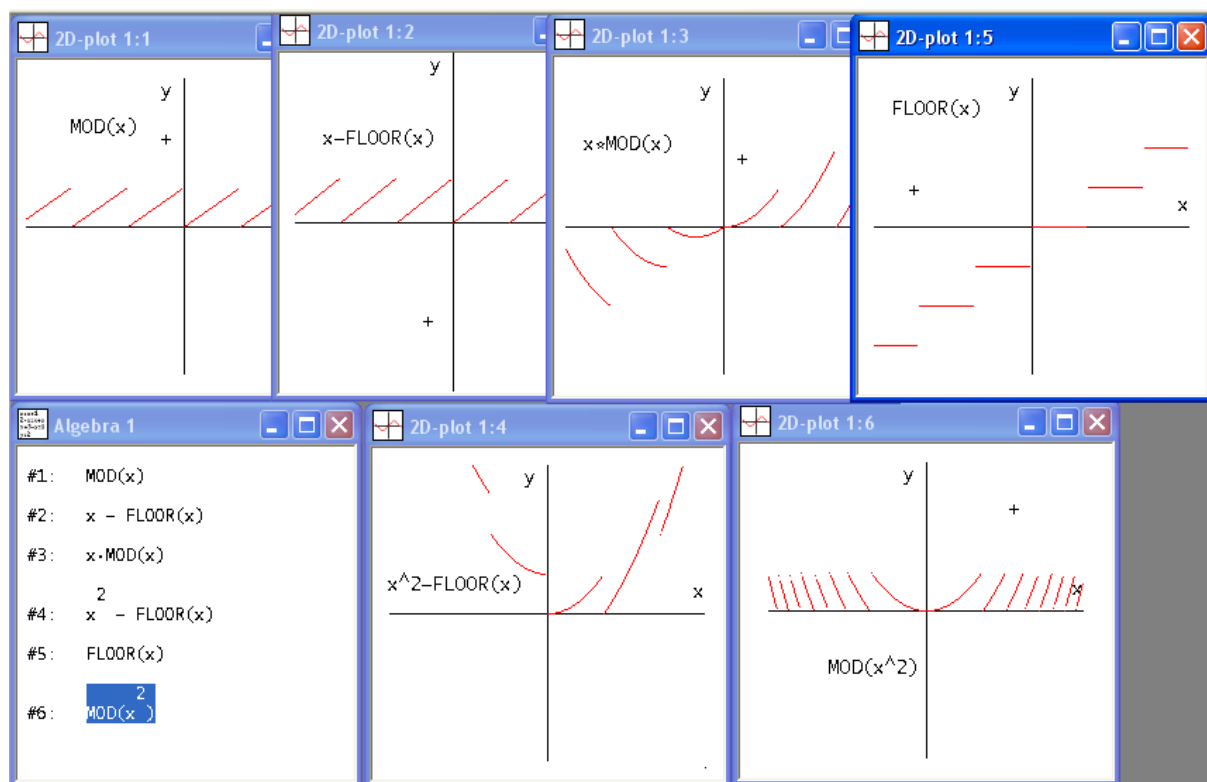


Figure 2

### The usage of $\text{MOD}(x)$ in plotting periodic functions.

To understand how  $\text{MOD}(x,y)$  can be used to plot  $T$ -periodic functions, we must understand how  $\text{MOD}(x,y)$  works (see window 4 of figure 2 for a plot of  $\text{MOD}(x)$ ).

$\text{MOD}(x)$  can be used to make sure that for instance  $f(3.2)$  will be reduced to  $f(0.2)$ . This means that  $f(x)$  can be 1-periodic: the part on the domain  $[0,1]$  is repeated.

$\text{MOD}(x)$  can be used to draw other, random composed, periodic functions.  $\text{MOD}(x)$  has period 1: every  $x$  is reduced to a value between 0 and 1. Replacing  $x$  with  $\text{MOD}(x)$  gives us a 1-periodic function!

A nice way to create a  $T$ -periodic functions is to transform the function defined on  $[0,1]$  by multiplication / translation etc.

An example:

We want a “periodic parabola” through the points  $(0,1)$  and  $(1,1)$  which is symmetrical on the axes  $x = 0.5$ . (see figure 3).

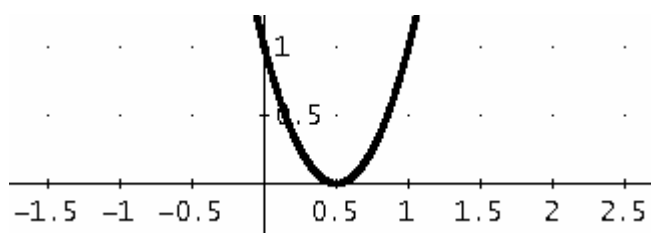


Figure 3: the quadratic curve.

First we must find the formula of the quadratic curve:  $f(x) := 4(x - 0.5)^2$ . This function surely is defined on  $[0, 1]$ . Now we replace  $x$  by  $\text{MOD}(x)$  which gives us the function  $f(x) := 4(\text{MOD}(x) - 0.5)^2$ . Plot the formula and you get figure 4.

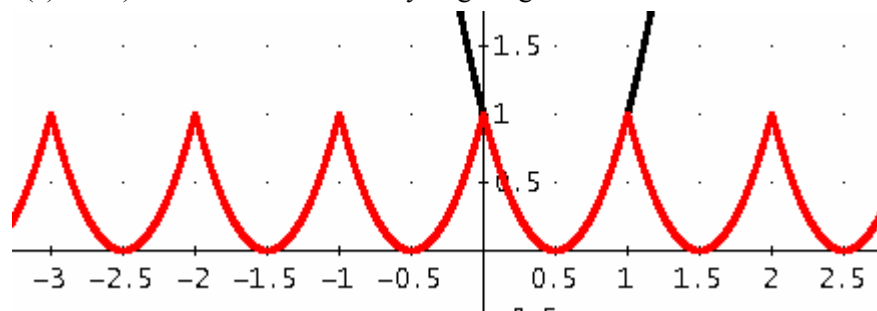


Figure 4: 2-periodic quadratic curve.

Suppose we want to translate this function so that the minimum of the quadratic curve occurs for  $x = 0$ , then we must replace  $x$  by  $x - 0.5$ . In the same way we can get a 2-periodic function by replacing  $x$  by  $x/2$  etc.

Another example:

We want a series of half-circles. At first you can define a half circle on  $[0, 1]$  (see figure 5). Solving for  $y$  gives us 2 possibilities, we choose the positive one.

The only thing we must do now is to replace  $x$  by  $\text{MOD}(x)$ . In the same way we get a 2-periodic series by replacing  $x$  by  $x/2$ . The period is twice as big but not the amplitude. So we'll have to multiply our function as well by a factor 2. Now we have a true 2-periodic series of half-circles.

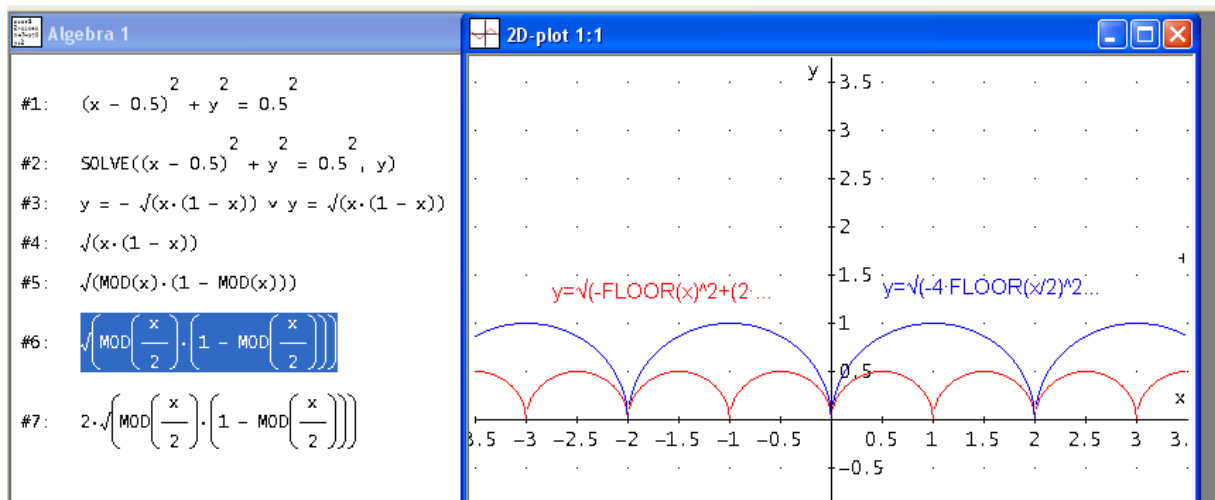


Figure 5: a series of half-circles

(It is interesting that DERIVE “simplifies” the functions, when we have them annotated in the plot window! Josef)

Noting that  $\text{MOD}(x, 2)$  already has a period 2, it must be possible to do things more easy.

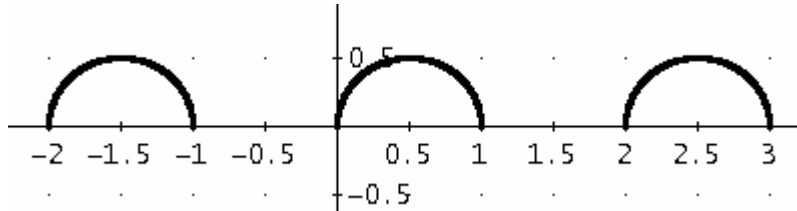
Suppose we plot  $y = \sqrt{x} \sqrt{1-x}$ .

Suppose we plot  $y = \sqrt{x} \sqrt{1-x}$ .

To make sure that the next period is produced we must be sure that  $x = 2.1$  for instance is reduced to  $x = 0.1$ . This can be done by  $\text{MOD}(x, 2)$ . So we replace  $x$  by  $\text{MOD}(x, 2)$  and get:

$$y = \sqrt{\text{MOD}(x, 2)} \sqrt{1 - \text{MOD}(x, 2)}.$$

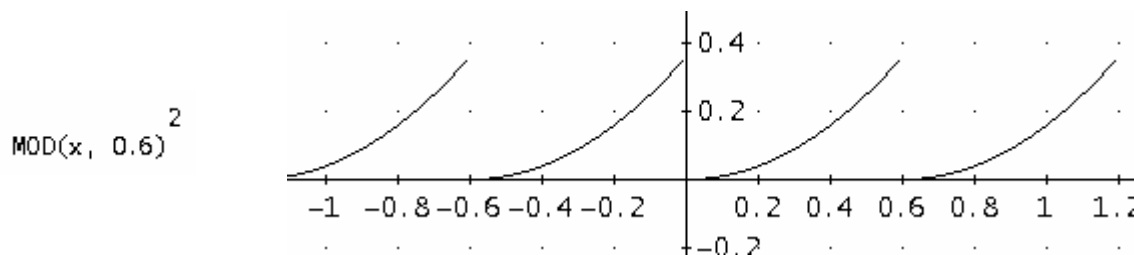
When we plot this function we get 2-periodic function in the next figure.



Let's try this on another function. Suppose  $f(x)$  is defined as:

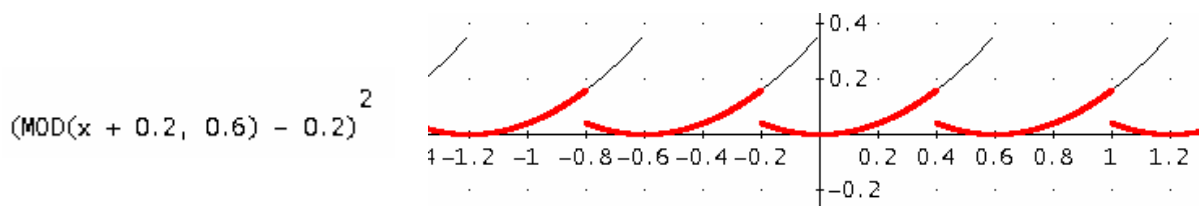
$f(x) = x^2$  when  $x \in [-0.2, 0.4]$  and 0.6 - periodic.

We are tempted to replace  $x$  by  $\text{MOD}(x, 0.6)$ . Why does this not work?



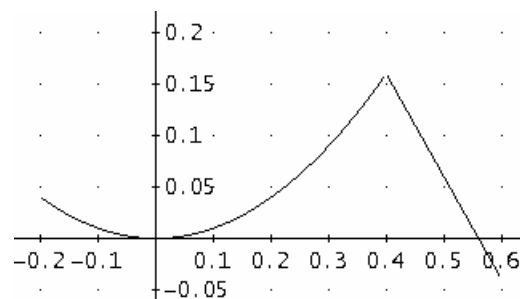
Looking at  $x = 0.7$  we see that  $\text{MOD}(x, 0.6)$  reduces it to 0.1 while it should be reduced to -0.1. We have to transform  $\text{MOD}(x)$ . The period is all right: 0.6. The starting point is not correct. It should be -0.2, so: will  $\text{MOD}(x + 0.2, 0.6)$  work? Not yet. Suppose we look again at  $x = 0.7$ . Now it is reduced to  $\text{MOD}(0.7 + 0.3, 0.6) = 0.3$ , still not correct. We must realize that we should subtract 0.2, so it should be:  $\text{MOD}(x + 0.2, 0.6) - 0.2$ !!!

We can generalize this to the procedure shown in the next figure:



## Combining ....

There is no restriction on the form of the function which is to be made periodic. We can join  $x^2$  on  $[-0.2, 0.4]$  with  $0.56 - x$  on  $[0.4, 0.6]$  for instance. In this case we make use of the IF- or CHI-function:

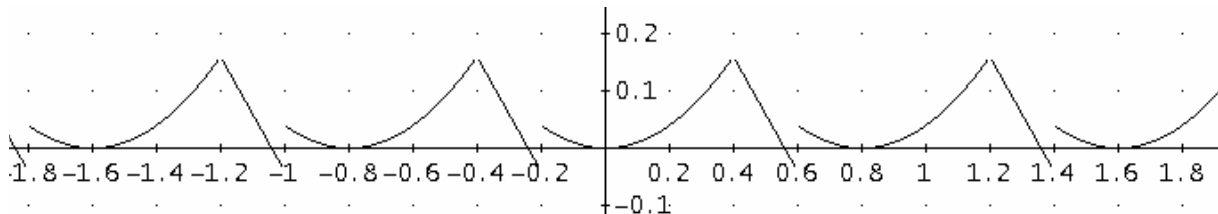


We can generalize this to the procedure shown in the next figure:

$$x^2 \cdot \chi(-0.2, x, 0.4) + \chi(0.4, x, 0.6) \cdot (-x + 0.56)$$

Substitute  $x$  by  $\text{MOD}(x+0.2, 0.8)-0.2$  and plot the resulting expression!

$$(\text{MOD}(x + 0.2, 0.8) - 0.2)^2 \cdot \chi(-0.2, \text{MOD}(x + 0.2, 0.8) - 0.2, 0.4) + \chi(0.4, \text{MOD}(x + 0.2, 0.8) - 0.2, 0.6) \cdot (- (\text{MOD}(x + 0.2, 0.8) - 0.2) + 0.56)$$

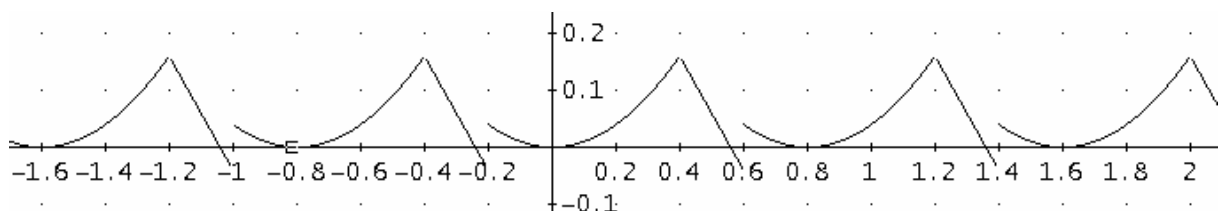


This works also with an IF-construction instead of the CHI-function for the piecewise defined function. Josef

$$\text{IF}(-0.2 \leq x \leq 0.4, x^2, \text{IF}(x \leq 0.6, 0.56 - x, ?))$$

$$\text{IF}(-0.2 \leq \text{MOD}(x + 0.2, 0.8) - 0.2 \leq 0.4, (\text{MOD}(x + 0.2, 0.8) - 0.2)^2,$$

$$\text{IF}(\text{MOD}(x + 0.2, 0.8) - 0.2 \leq 0.6, 0.56 - (\text{MOD}(x + 0.2, 0.8) - 0.2), ?))$$



## How to automate the process.

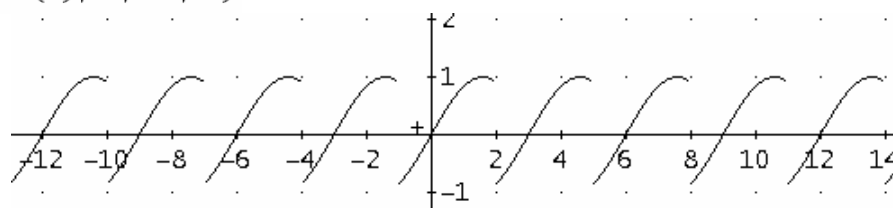
In lectures on Fourier analysis students construct a function  $f(t)$  by summing harmonics.

When the original function is  $T$ -periodic, they can match it with their own constructed function. To avoid unnecessary arithmetic one could automate the process by defining the next function:

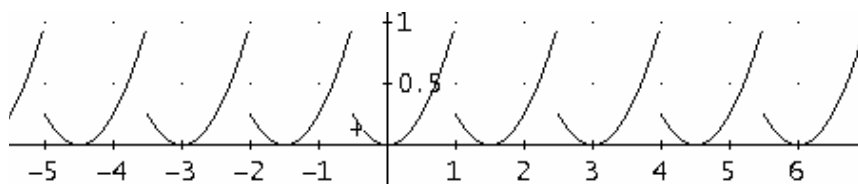
$$\text{PERIODIC}(f, t, a, b) := \lim_{t \rightarrow \text{MOD}(t - a, b - a) + a} f$$

From now on it's rather easy to use.

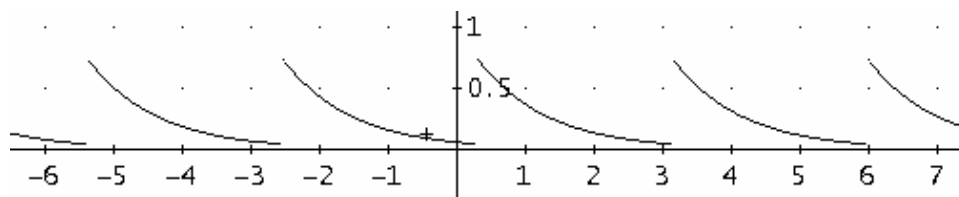
$$\text{PERIODIC}(\text{SIN}(t), t, -1, 2)$$



$\text{PERIODIC}(t^2, t, -0.5, 1)$



$\text{PERIODIC}(\exp(-t), t, 0.3, \pi)$

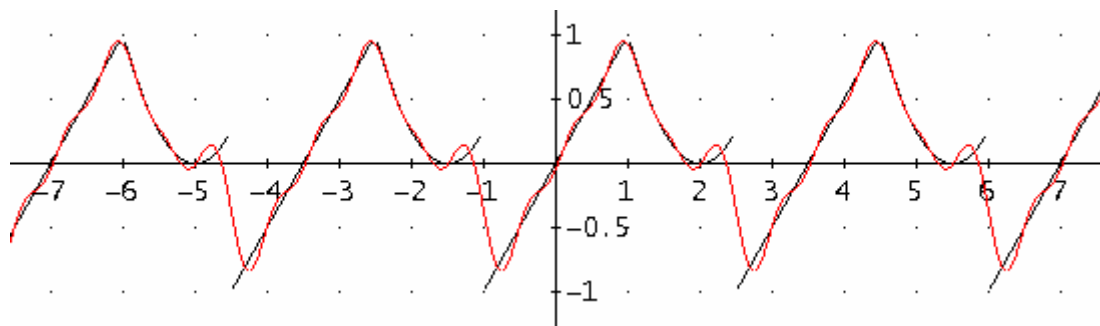


We have now the possibility to compare a Fourier - approximation of  $f(x)$  with  $f(x)$  itself, as you can see below!

$F(t) := \chi(-1, t, 1) \cdot t + \chi(1, t, 2.5) \cdot (t - 2)^2$

$\text{PERIODIC}(F(t), t, -1, 2.5)$

$\text{FOURIER}(F(t), t, -1, 2.5, 5)$



## Conclusion.

DERIVE can be a useful tool for plotting T-periodic functions. Although it is possible to define and use T-periodic functions with CHI and SUM they can't be plotted. The use of MOD( $x, T$ ) is simpler and more elegant. Nevertheless, the process of defining true T-periodic functions with CHI and SUM and the use of transformations is very learningful. The use of PERIODIC() appears to be very useful in the classroom.

## DERIVE, Newton Method and Ill - Conditioned Problems

Jose M.Cardia Lopes, Porto, Portugal

J. Böhm (1993) has recently written in this magazine about the Newton method for solving non linear equations pointing out, by means of an interesting example, the crucial importance of the choice of the initial values. The popularity of this method is due mostly to its quadratic convergence if the initial values are well chosen. (If it doesn't converge the divergence soon becomes obvious.) Nevertheless with ill-conditioned problems the Newton method (as any method ...) can lead to a large number of iterations and to not very good solutions.

A particular class off ill-conditioned problems which often appear in the classroom is the problem of solving polynomial equations with multiple roots. In fact it can be demonstrated that if we have an equation  $f(x) = 0$  with a multiple root  $X$ , if  $f^p(x)$  is the computer representation of  $f(x)$  (with the "perturbations" due to the ill-condition), if  $\Delta$  is an upper bound on the perturbations in  $f(x)$ , and if  $X^p$  is the corresponding root of  $f^p(x) = 0$ , then

$$|X^p - X| \approx \left( \frac{\Delta m!}{|f^{(m)}(X)|} \right)^{\frac{1}{m}}$$

where  $m$  is the multiplicity of the root. With this expression we can evaluate the effect of a perturbation in  $f(x)$  on the determination of the multiple root of the equation. As a result of the power  $1/m$  the determination of multiple roots may be an ill-conditioned problem.

Since the ill-conditioning is a consequence of the problem structure, we can only avoid it if we change the problem structure. For a polynomial equation we can substitute the evaluation of the multiple root of

$$f(x) = 0$$

by the evaluation of the single root of

$$u(x) = 0$$

where

$$u(x) = \frac{f(x)}{f'(x)}$$

(see for example, Chapra e Canale, 1990).

This is another situation in which DERIVE can be a very good pedagogical tool. We can graphically show how the structural change of the problem improves dramatically the path to the root. We can see some examples:

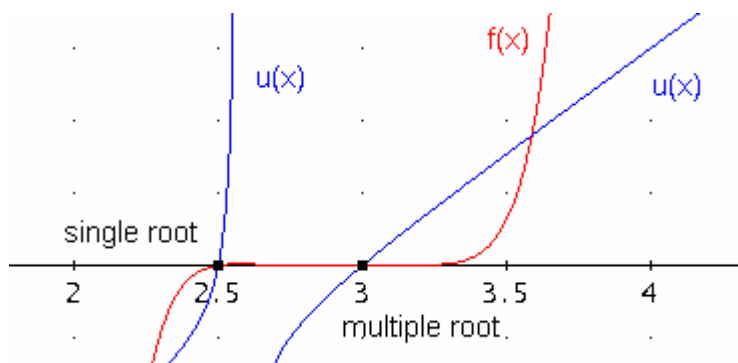
Example 1:  $f(x) = (x - 3)^6 (2x - 5)$

Derive has no difficulty to solve this equation but the graphical representation suggests that the Newton method converges on the multiple root  $x = 3$  at a very slow rate (notice that the function does not cross the  $x$ -axis at  $x = 3$ ). The situation is very different with  $u(x)$ .

Example 1:

#1:  $f(x) := (x - 3)^6 \cdot (2 \cdot x - 5)$

#2:  $u(x) := \frac{f(x)}{\frac{d}{dx} f(x)}$

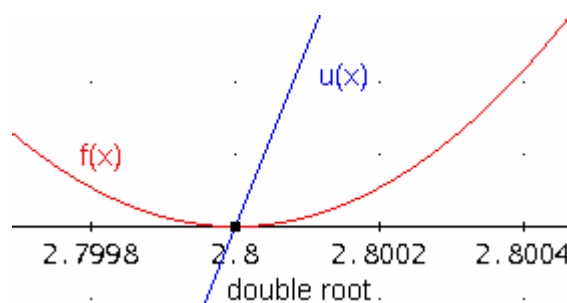


Example 2:  $f(x) = (5x - 14)^2 (x^2 + 3)$

Example 2:

#3:  $f(x) := (5 \cdot x - 14)^2 \cdot (x^2 + 3)$

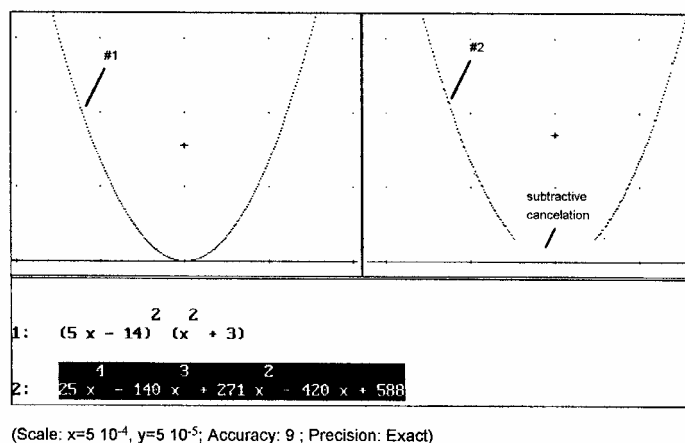
#4:  $u(x) := \frac{f(x)}{\frac{d}{dx} f(x)}$



If the lecture is designed for beginners there will probably be some interest in a more detailed approach (like the one proposed by Dyer (1993)) to show the results of some iterations.

In addition the second example points out the importance of the manipulation of the mathematical expressions in order to minimize the number of arithmetic operations. See the plots of the factorized and the expanded function.

(There is no difference now in higher versions of DERIVE, Josef)



Screen Shot from the 1994 DERIVE for DOS version

## References

- Böhm J. (1993) "Newton-Raphson's Chaos", The DERIVE Newsletter #12, p 10 -13  
 Chapra, S.C & Canale, R.P. (1990), "Numerical Methods for Engineers", 2<sup>nd</sup> ed, Mc-Graw-Hill  
 Dyer, D. (1993) "The Bisection Method and DERIVE", The DERIVE Newsletter #11, p 11-13



**Find here a worksheet of Keith Eames:**

**Exercise**

Use the graphical capability of *Derive* to answer these questions. Check your answers algebraically using *Derive*.

Explain how you can get from  $F(x)$ , by a single transformation whenever possible, to each  $G(x)$  in all of the following questions:

1. Draw the graph of  $F(x) = x^2 - 2x + 3$ .

a)  $G(x) = x^2 + 2x + 3$

b)  $G(x) = -x^2 + 2x - 3$

c)  $G(x) = -x^2 - 2x - 3$

d)  $G(x) = x^2 + 4x + 6$

e)  $G(x) = 3x^2 - 6x + 9$

c)  $G(x) = 4x^2 - 4x + 3$

2. Draw the graph of  $F(x) = (2x - 1)(3x + 5)$

a)  $G(x) = 6x^2 + 7x + 4$

b)  $G(x) = 6x^2 - 7x - 5$

c)  $G(x) = 6x^2 - 29x - 28$

d)  $G(x) = 10 - 14x - 12x^2$

e)  $G(x) = 54x^2 - 21x - 5$

f)  $G(x) = 6x^2 + 31x + 28$

3. Draw the graph of  $F(x) = 2x^3 - 3x^2 - 9x + 10$

a)  $G(x) = 2x^3 - 3x^2 - 9x$

b)  $G(x) = 2x^3 - 9x^2 + 3x + 14$

c)  $G(x) = -4x^3 + 6x^2 + 18x - 20$

d)  $G(x) = 10 + 18x - 12x^2 - 16x^3$

e)  $G(x) = 2x^3 + 9x^2 + 3x$

f)  $G(x) = 16x^3 - 12x^2 - 18x - 10$

4. Draw the graph of  $F(x) = \sin(x)$

a)  $G(x) = 2 \sin(x) \cos(x)$

b)  $G(x) = \frac{\sqrt{3}}{2} \sin(x) + \frac{1}{2} \cos(x)$

c)  $G(x) = \cos$

d)  $G(x) = \sqrt{2} \sin(x) - \sqrt{2} \cos(x)$

e)  $G(x) = \sin(x) + \cos(x)$

f)  $G(x) = 1 + 3 \sin(x) - 4 \sin^3 x$

All the trig functions in the above are in radians not degrees

**Hint: Use the well known fractional multiples of  $\pi$**

5. Is it always possible to map a quadratic function to another quadratic function by a series of transformations? Experiment with your own quadratic equations. Can you prove any of your findings? Give some examples! What is the maximum number of different single transformations that is required?

|     |  |          |
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| p40 | Keith Eames: Functions Transformations | D-N-L#16 |
|-----|--|----------|

## Answers

1. Use a trial and error method to find possible answers.

**Simplify**  $F(x+1)$

Compare the given answers with  $G(x)$

a)  $F(-x)$       b)  $-F(x)$       c)  $-F(-x)$       d)  $F(x+3)$       e)  $3 F(x)$       f)  $F(2x)$

2. If **Derive** does not give the answer shown **Expand** the expressions and compare.

a)  $F(x)+9$       b)  $F(-x)$       c)  $F(x-3)$       d)  $-2 F(x)$       e)  $F(-3x)$       f)  $F(x+2)-5$

3.

a)  $F(x) - 10$       b)  $F(x-1)$       c)  $-2 F(x)$       d)  $F(-2x)$       e)  $F(x+2)+4$       f)  $F(2x)-20$

4.

a)  $F(2x)$       b)  $F(x + \pi/6)$       c)  $F(2x)$       d)  $2 F(x - \pi/4)$       e)  $\sqrt{2}F(x + \pi/4)$       f)  $F(3x) + 1$

5.  $F(x) = ax^2 + bx + c$  can be transformed to  $G(x) = rx^2 + sx + t$  by at most 4 transformations, this depends on the values of  $a$  and  $r$ .

If  $r$  and  $a$  are both positive then it is 3 transformations:

$$F\left(\frac{\sqrt{r}}{\sqrt{a}}\left(x - \frac{b}{2\sqrt{r}\sqrt{a}} + \frac{s}{2r}\right)\right) + t - c - \frac{s^2}{4r} + \frac{b^2}{4a}$$

How does this formula change if  $a$  is positive and  $r$  is negative?

*When you have finished this assignment and you are happy if your results and conclusions ask for the next worksheet.*

**Answer the questions on it without using Derive.**

A correction: In the last DNL I published H. Scheuermann's phone number. One digit was wrong, this is the right number: 06192 27568

*Please don't forget to renew your subscription for 1995 in time. Administration would be much easier for us. If you don't cancel then we assume that you will stay a member. Many thanks for your cooperation.*

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