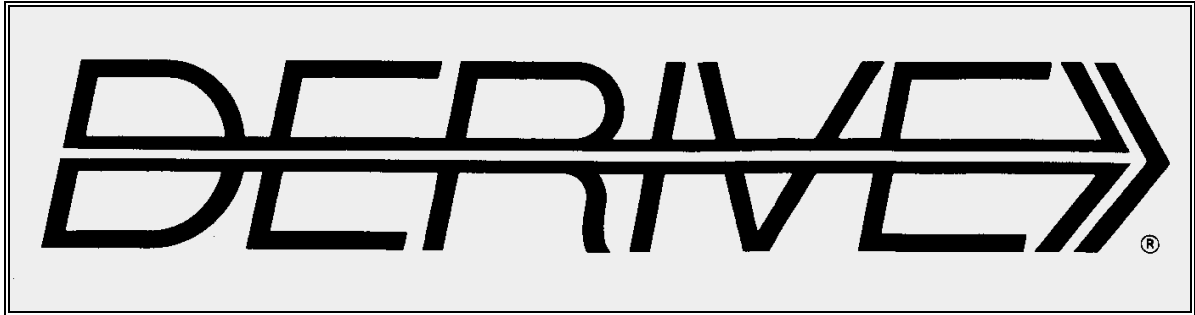


**THE BULLETIN OF THE**



**USER GROUP**

**+ CAS-TI**

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### **Proceedings of former DERIVE & CAS-TI and ACDCA-Conferences**

You are invited to browse and to download lectures and workshops (all in df-format) of (almost) all earlier Conferences starting with Krems 1992 and ending with Malaga 2010:

<http://rfdz.ph-noe.ac.at/acdca/konferenzen.html>

Krems 1992, Krems 1993, Derive Days Düsseldorf 1995, Bonn 1996, Gettysburg 1998, Portoroz 2000, Liverpool 2000, Vienna 2002, Montreal 2004, Dresden 2006, Buffelspoort (South Africa) 2008, Malaga 2010

Some proceedings have been published originally in printed form only. I scanned them all and interlinked them with the respective contents page. Many thanks to Peter Hofbauer who set all files on the web.

Happy Browsing and Happy Remembering for many of you -

- and don't forget:

**Attend TIME2012 in Tartu, Estonia**  
**July 11-14, 2012**  
<http://time2012.ut.ee/>

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**Browse a library of over 2,400 educational videos at**

<http://www.khanacademy.org>

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**Nur in deutscher Sprache (in German only):**

*Franz Embachers* Mathe-Online wurde erweitert und verbessert:

<http://www.mathe-online.at/neu.html>

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DUG-Mitglied *Johannes Zerbs* ist sehr aktiv. Gerne veröffentliche ich den Link zu seinen Publikationen und wünsche weiterhin gutes Gelingen und viel Erfolg.

**MATHE CHILLED –ONLINE - ein Buch, eine Sammlung, ein "Rätselblock" von Übungschilis(-kapiteln) nach den neuen Standards und Kompetenzen der Unterstufe**

[http://www.johanneszerbs.at/visionen/index.php?option=com\\_docman&task=cat\\_view&gid=41&Itemid=12](http://www.johanneszerbs.at/visionen/index.php?option=com_docman&task=cat_view&gid=41&Itemid=12)

Alle Infos mit der Buchbeschreibung unter Klick auf:

→ [www.johanneszerbs.at](http://www.johanneszerbs.at) → linker Menüpunkt „Online-Mathe-Übungscenter“ →  
 → UE-klasse-center → UE-klasse-Worddokus

Von der Homepage gelangt man auch zu einer mathematischen „Millionenshow“ und zu einem Mathe-Quiz, Josef

Liebe DUG-Mitglieder,  
eigentlich wollte ich in diesem Newsletter vor allem die Beiträge von Dietmar Oertel, Piotr Trebisz und Duncan McDougall weiter bzw. zu Ende führen. Ich war auch schon ziemlich weit mit den Übersetzungen der deutschen Beiträge, aber dann erreichten mich einige Anfragen und mails, die schließlich über weitere Kommunikation eine so heftige Eigendynamik entwickelten, dass ich sie nicht länger liegen lassen konnte. Nur Robert Setif's Anfrage musste auf den nächsten DNL verschoben werden.

Ich möchte Sie besonders darauf hinweisen, dass nun die Tagungsbände aller DERIVE- und ACDCA-Konferenzen im pdf-Format verfügbar sind. Beachten Sie bitte den Hinweis auf der Info-Seite.

Ich freue mich, wieder einen TI-Nspire-Beitrag vorstellen zu können. Mit Version 3 kann man sehr ordentlich arbeiten. Nun sind wirklich schon einige von uns lange gewünschten DERIVE-Möglichkeiten implementiert worden.

Besonders spannend für mich war die Beschäftigung mit der konformen Abbildung bzw. deren Umkehrung, zu der mich ein Brief von Klaus Körner gebracht hat.

Ich habe auch sehr wenig Ahnung über Tensoren gehabt. Dank Tom Fowler weiß ich nun ein wenig mehr darüber. Roger Folsom schreibt einiges über „Comparative Statics“ - wofür ich keine passende deutsche Übersetzung finden konnte. Ich habe aber ein paar sehr schöne Unterlagen im Web gefunden, die zu einer weniger bekannten Anwendung der Differentialrechnung im wirtschaftlichen Bereich führen.

Walter Klinger ist wieder zum Einsatz von DERIVE im Unterricht zurückgekehrt und erlebt mit der Version 6 immer wieder Überraschungen, die er gerne mit mir bespricht. Walter, vielen Dank für die anregenden Gespräche.

Dear DUG Members,  
Actually I intended to continue the contributions of Dietmar Oertel, Piotr Trebisz and Duncan McDougall in this newsletter. I was very busy translating the German papers but then I received some requests and emails which together with the resulting communication got a strong self dynamics and I could not leave them on my to-do-list any longer. Only Robert Setif's request must be left for the next issue.

I'd like to inform you that the proceedings of all earlier DERIVE and ACDCA Conferences are available in pdf-format and can be downloaded. More info on the information page.

I am very pleased about being able to present an TI-Nspire related article again. Version 3 allows comfortable and powerful working. Finally some of long desired DERIVE features have been implemented

Extra exciting for me was dealing with the conformal mapping and its inverse provoked by Klaus Körner's mail.

I must admit that I knew nothing whatsoever about tensors. Thanks Tom Fowler I now have at least a weak idea about these mathematical objects. Roger Folsom writes about "Comparative Statics" - I couldn't find an appropriate German translation. What I could find are some fine resources on the web. They are informing about a not so well known application of calculus in economics.

Walter Klinger returned to use DERIVE in secondary level math teaching. He is coming across some surprises with DERIVE 6 which he likes to discuss with me. Many thanks for the inspiring talks.

With my best regards

Mit herzlichen Grüßen



Download all DNL-DERIVE- and TI-files from

<http://www.austromath.at/dug/>

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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### Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue: December 2011

### **Preview: Contributions waiting to be published**

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER  
Wonderful World of Pedal Curves, J. Böhm, AUT  
Tools for 3D-Problems, P. Lüke-Rosendahl, GER  
Hill-Encryption, J. Böhm, AUT  
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT  
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT  
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER  
Overcoming Branch & Bound by Simulation, J. Böhm, AUT  
Graphics World, Currency Change, P. Charland, CAN  
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT  
Logos of Companies as an Inspiration for Math Teaching  
Exciting Surfaces in the FAZ / Pierre Charland's Graphics Gallery  
BooleanPlots.mth, P. Schofield, UK  
Old traditional examples for a CAS – what's new? J. Böhm, AUT  
Truth Tables on the TI, M. R. Phillips, USA  
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA  
Embroidery Patterns, H. Ludwig, GER  
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ & Rob Gough, UK  
Tutorials for the NSpireCAS, G. Herweyers, BEL  
Some Projects with Students, R. Schröder, GER  
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA  
Treating Differential Equations (M. Beaudin, G. Piccard, Ch. Trottier), CAN  
A New Approach to Taylor Series, D. Oertel, GER  
Statistics with TI-Nspire, G. Herweyers, BEL  
Cesar Multiplication, G. Schödl, AUT  
Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT  
Rational Hooks, J. Lechner, AUT  
Cubus Simus, H. Ludwig, GER  
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT  
and others

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# On Quadratics and Conic Sections<sup>1</sup>

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Nowadays conic sections have nearly disappeared from textbooks. Nevertheless it is an interesting field containing a lot of nontrivial algebra.

## 1. (General 2-Variable) Quadratic Equations

The intersection of a circular cone with the equation  $z^2 = x^2 + y^2$  in  $\mathbb{R}^3$  with a plane  $\alpha x + \beta y + \gamma z = \varepsilon$  yields a so called **quadratic** which only has terms in  $x^2$ ,  $y^2$ ,  $x \cdot y$ ,  $x$ ,  $y$  and a constant term. It looks like

$$(1) \quad a \cdot x^2 + 2b \cdot x \cdot y + c \cdot y^2 + 2d \cdot x + 2e \cdot y + f = 0.$$

Due to its (geometrical) genesis its graph is called **conic section** or just **conic**.

The left side of this equation is elegantly expressed by  $W^T \cdot M \cdot W$  with  $M$  a symmetric matrix

$M = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$  and  $W$  a vector  $W = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ . This construction leads to the factors 2 in

the coefficients of the mixed and the linear terms [CS, 2.2].

© the declarations:		
$w(x,y) := \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$		Done
$term(x,y) := ((w(x,y))^T \cdot m \cdot w(x,y)) [1 \ 1]$		Done
$eqn(x,y) := term(x,y) = 0$		Done
© for the examples we must declare a specific matrix M		
$m := \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$	$expand(eqn(x,y))$ : © 2nd degree polynomial in x and y,	$a \cdot x^2 + 2 \cdot b \cdot x \cdot y + 2 \cdot d \cdot x + c \cdot y^2 + 2 \cdot e \cdot y + f = 0$
$m := \begin{bmatrix} a & 0 & d \\ 0 & 0 & e \\ d & e & f \end{bmatrix}$	$term(x,y)$ : © a general expression for a parabola,	$a \cdot x^2 + 2 \cdot d \cdot x + 2 \cdot e \cdot y + f$
© now we become specific:		
$m := \begin{bmatrix} 9 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -225 \end{bmatrix}$	$propFrac\left(\frac{eqn(x,y)}{225}\right) + 1$ : © an ellipsi,	$\frac{x^2}{25} + \frac{y^2}{9} = 1$
$m := \begin{bmatrix} 4 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$factor(eqn(x,y))$ : © a pair of straight line,	$(2 \cdot x - 3 \cdot y) \cdot (2 \cdot x + 3 \cdot y) = 0$
$m := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{bmatrix}$	$eqn(x,y) + x^2 + 1$ : © a parabola	$y = x^2 + 1$

<sup>1</sup> With this contribution is connected an extended TI-Nspire document *OnConicSections\_NN.tns* (NN is 20 or 30 depending on which release of TI-Nspire CAS it should run. 20 is for release 2.x and 30 for the latest release 3.x). When referring to this document in this contribution we denote this by [CS, Pr.Pg] where Pr means problem and Pg the page in this problem.

## 2. Classification of conic sections

At first we will investigate the case  $b = d = e = 0$ . Equation (1) is reduced to

$$(2) \quad a \cdot x^2 + c \cdot y^2 + f = 0.$$

In case  $a \cdot c \cdot f \neq 0$  the classification is done easily by analyzing the signs of these coefficients.

If  $a$ ,  $c$  and  $f$  have equal signs then the solution set is empty. Otherwise we have ellipses or hyperbolas, so called central curves with a centre point at the origin of the coordinate system and their half-axes on the coordinate axes.

$a$	$c$	$f$	Type
$\pm$	$\pm$	$\mp$	Ellipse
$\pm$	$\mp$	$\pm$	Hyperbola, symm. wrt $y$ -axis
$\mp$	$\pm$	$\pm$	Hyperbola, symm. wrt $x$ -axis
$\pm$	$\pm$	$\pm$	Empty set

Table 1

If however  $a \cdot c \cdot f = 0$ , - i.e. at least one of the coefficients disappears -, then we have 3 instances:

### 1. Pair of straight lines (exactly one coefficient is zero):

If  $f = 0$  and  $a$  and  $c$  have different signs, the left side of equation (2) decomposes into two linear factors. This results in two symmetric straight lines through the origin.

If  $a = 0$  or  $c = 0$  we get a pair of parallels with regard to the  $x$ - or the  $y$ -axis, if the signs of  $a$  or  $c$  and  $f$  are different.

### 2. Double line (exactly two coefficients are zero):

In case  $a$  and  $f$  or  $c$  and  $f$  disappear then the  $x$ -axis or the  $y$ -axis is the only one solution of the corresponding equation.

### 3. Point:

If  $f = 0$  and  $a$  and  $c$  have equal signs the equation from above is only fulfilled by the origin.

In all other cases there will be no more genuine intersection figure, i.e. the solution set is empty. Except if  $a = c = f = 0$ . Then equation (2) is fulfilled by the whole  $xy$ -plane.

The following table shows this in a clear form:

$a$	$c$	$f$	Type
$\pm$	$\mp$	0	Pair of straight lines through the origin
0	$\pm$	$\mp$	Pair of parallel lines with respect to x-axis
$\pm$	0	$\mp$	Pair of parallel lines with respect to y-axis
$\pm$	0	0	Double line, the x-axis
0	$\pm$	0	Double line, the y-axis
$\pm$	$\pm$	0	Point, the origin
$\pm$	0	$\pm$	Empty set
0	$\pm$	$\pm$	Empty set
0	0	$\pm$	Empty set
0	0	0	Whole xy-plane

Table 2

### The impact of coefficients $b$ , $d$ and $e$ [CS, 2.3 and 2.4]

If just  $b \neq 0$  (and further on  $d = e = 0$ ) then a rotation where  $x$  is replaced by  $x \cdot \cos(\theta) - y \cdot \sin(\theta)$  and  $y$  is replaced by  $x \cdot \sin(\theta) + y \cdot \cos(\theta)$  can bring the coefficient of the mixed term to zero without introducing linear terms in  $x$  or  $y$ . This means that we have again one of the previous cases.

System Matrix syst:
$$\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

Done

© translation to remove linear terms\*

$koeff(term(x+u, y+v))$

Coefficient Matrix koef:
$$\begin{bmatrix} x^2 & x \cdot y & y^2 & x & y & const \\ a & 2 \cdot b & c & 2 \cdot (a \cdot u + b \cdot v + d) & 2 \cdot (b \cdot u + c \cdot v + e) & a \cdot u^2 + 2 \cdot u \cdot (b \cdot v + d) + c \cdot v^2 + 2 \cdot e \cdot v + f \end{bmatrix}$$

System Matrix syst:
$$\begin{bmatrix} a & b & a \cdot u + b \cdot v + d \\ b & c & b \cdot u + c \cdot v + e \\ a \cdot u + b \cdot v + d & b \cdot u + c \cdot v + e & a \cdot u^2 + 2 \cdot u \cdot (b \cdot v + d) + c \cdot v^2 + 2 \cdot e \cdot v + f \end{bmatrix}$$

Done

$solve\left(\begin{cases} syst[1,3]=0 \\ syst[2,3]=0 \end{cases}, u, v\right)$

$u = \frac{b \cdot e - c \cdot d}{a \cdot c - b^2}$  and  $v = \frac{-(a \cdot e - b \cdot d)}{a \cdot c - b^2}$

$koeff(term(x+u, y+v)) u = \frac{b \cdot e - c \cdot d}{a \cdot c - b^2}$  and  $v = \frac{-(a \cdot e - b \cdot d)}{a \cdot c - b^2}$

Coefficient Matrix koef:
$$\begin{bmatrix} x^2 & x \cdot y & y^2 & x & y & const \\ a & 2 \cdot b & c & 0 & 0 & \frac{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2}{a \cdot c - b^2} \end{bmatrix}$$

System Matrix syst:
$$\begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & \frac{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2}{a \cdot c - b^2} \end{bmatrix}$$

Done

© Result: A translation with  $u = \frac{b \cdot e - c \cdot d}{a \cdot c - b^2}$  and  $v = \frac{b \cdot d - a \cdot e}{a \cdot c - b^2}$  lets the coefficients of  $x$  and  $y$  disappear.

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The condition for removing the mixed term is  $\tan(2\theta) = \frac{2b}{a-c}$ . (In case  $a = c$  the conic section is a circle which is invariant with respect to rotations around its centre.)

Likewise (at an arbitrary  $b$ ) and  $d, e \neq 0$  the translation  $x := x + u$  and  $y := y + v$  may remove the coefficients of the linear terms in  $x$  and  $y$ . The conditions are  $u = \frac{b \cdot e - c \cdot d}{a \cdot c - b^2}$  and  $v = \frac{b \cdot d - a \cdot e}{a \cdot c - b^2}$ . The coefficients of  $x^2$ ,  $x \cdot y$  and  $y^2$  remain unchanged.

© Here we determine a rotation that the coefficient of  $x \cdot y$  disappear\*

© first we assign syst (from page 3) to m:

m:=syst

$$\begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & \frac{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2}{a \cdot c - b^2} \end{bmatrix}$$

© Statement to calculate the angle of rotation\*

coeff(term(x\*cos(theta)-y\*sin(theta),x\*sin(theta)+y\*cos(theta)))

coefficient Matrix koef:

$x^2$	$x \cdot y$	$y^2$	$x \cdot y$	const
$a \cdot (\cos(\theta))^2 + (2 \cdot b \cdot \cos(\theta) + c \cdot \sin(\theta)) \cdot \sin(\theta)$	$-2 \cdot (a \cdot \sin(\theta) \cdot \cos(\theta) - b \cdot (2 \cdot (\cos(\theta))^2 - 1) - c \cdot \sin(\theta) \cdot \cos(\theta))$	$a \cdot (\sin(\theta))^2 - (2 \cdot b \cdot \sin(\theta) - c \cdot \cos(\theta)) \cdot \cos(\theta)$	0	$\frac{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2}{a \cdot c - b^2}$

system Matrix syst:

$$\begin{bmatrix} a \cdot (\cos(\theta))^2 + (2 \cdot b \cdot \cos(\theta) + c \cdot \sin(\theta)) \cdot \sin(\theta) & -2 \cdot (a \cdot \sin(\theta) \cdot \cos(\theta) - b \cdot (2 \cdot (\cos(\theta))^2 - 1) - c \cdot \sin(\theta) \cdot \cos(\theta)) & a \cdot (\sin(\theta))^2 - (2 \cdot b \cdot \sin(\theta) - c \cdot \cos(\theta)) \cdot \cos(\theta) & 0 \\ -2 \cdot (a \cdot \sin(\theta) \cdot \cos(\theta) - b \cdot (2 \cdot (\cos(\theta))^2 - 1) - c \cdot \sin(\theta) \cdot \cos(\theta)) & a \cdot (\sin(\theta))^2 - (2 \cdot b \cdot \sin(\theta) - c \cdot \cos(\theta)) \cdot \cos(\theta) & 0 & 0 \\ 0 & 0 & \frac{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2}{a \cdot c - b^2} & 0 \end{bmatrix}$$

Done

© now we have to calculate  $\theta$  that the mixed term disappears\*

solve(syst[1,2]=0,theta)

$$\theta = \frac{\tan^{-1}\left(\frac{2 \cdot b}{a - c}\right) + n \cdot \pi}{2}$$

© result: The condition for  $\theta$  is:  $\tan(2\theta) = \frac{2b}{a-c}$

© because m was not changed we can now take the statement from above and substitute \*

coeff(term(x\*cos(theta)-y\*sin(theta),x\*sin(theta)+y\*cos(theta))) $\theta = \frac{1}{2} \cdot \tan^{-1}\left(\frac{2 \cdot b}{a - c}\right)$

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There is a lot of algebra in the Calculator Window, isn't it?

As one easily can see, these conditions are only valid if  $a \cdot c - b^2 \neq 0$ . This case has not been discussed up to now. Here we have a parabola, as we will see in the next section.

### The parabola case [CS, 2.5]

If  $a \cdot c - b^2 = 0$  and  $a \neq 0$  we can omit  $c$  in equation (1) and we get

$$(1') \quad a \cdot x^2 + 2b \cdot x \cdot y + \frac{b^2}{a} \cdot y^2 + 2d \cdot x + 2e \cdot y + f = 0$$

This equation can be brought to the form

$$(3) \quad a' \cdot x^2 + 2e' \cdot y = 0$$

with  $a'$  and  $e'$  only depending on the coefficients of equation (1'). (If in (1') the term with  $x^2$  should be omitted then (3) takes the form  $c' \cdot y^2 + 2d' \cdot x = 0$ . The calculations are principally the same.)



In both cases we have parabolas, so called non-central curves. They are symmetric with respect to the  $y$ - or  $x$ -axis and they are passing the origin.

Applying a rotation by angle  $\theta$  with  $\tan(\theta) = \frac{b}{a}$  we can remove the mixed term and the term which is quadratic in  $y$  in equation (1') as well. So it takes the form

$$a' \cdot x^2 + 2d' \cdot x + 2e' \cdot y + f' = 0.$$

And after a translation  $x := x + u$  and  $y := y + v$  with  $u = -\frac{d'}{a'}$  and  $v = \frac{d'^2 - a' \cdot f'}{2a' \cdot e'}$  we finally yield equation (3).

**Example 2** (a parabola):

The terms of the parabola are called **t4**, **t5** and **t6** and the appropriate graphs are **f4(x)**, **f5(x)** and **f6(x)**.

New variables have been introduced to avoid side effects in the actual problem.

The entry line in the G&G-windows has been hidden by Ctrl+G.

© the original term and its graph

$$t4 := x^2 - 4 \cdot x \cdot y + 4 \cdot y^2 - 2 \cdot x - 6 \cdot y - 7 \quad x^2 + x \cdot (-4 \cdot y - 2) + 4 \cdot y^2 - 6 \cdot y - 7$$

$$f4(x) := \text{zeros}(x^2 + x \cdot (-4 \cdot y - 2) + 4 \cdot y^2 - 6 \cdot y - 7, y) \quad \text{Done}$$

© rotation first

$$t5 := \text{rotpol}(t4, \tan^{-1}(-2)) \quad 5 \cdot x^2 + 2 \cdot \sqrt{5} \cdot x - 2 \cdot \sqrt{5} \cdot y - 7$$

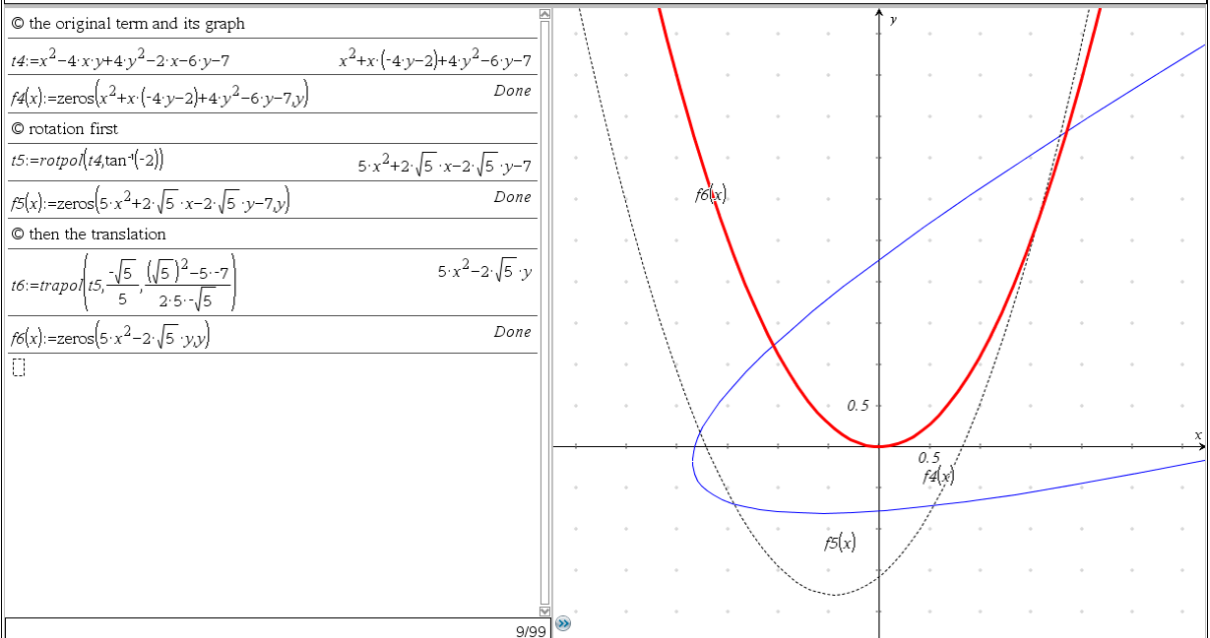
$$f5(x) := \text{zeros}(5 \cdot x^2 + 2 \cdot \sqrt{5} \cdot x - 2 \cdot \sqrt{5} \cdot y - 7, y) \quad \text{Done}$$

© then the translation

$$t6 := \text{trapol}\left(t5, \frac{\sqrt{5}}{5}, \frac{(\sqrt{5})^2 - 5 \cdot 7}{2 \cdot 5 \cdot \sqrt{5}}\right) \quad 5 \cdot x^2 - 2 \cdot \sqrt{5} \cdot y$$

$$f6(x) := \text{zeros}(5 \cdot x^2 - 2 \cdot \sqrt{5} \cdot y, y) \quad \text{Done}$$

□



## Conclusion

We can constrain the investigation of equation (1) to two cases:

It is sufficient that either  $b = d = e = 0$  or that  $b = c = d = f = 0$  (or  $a = b = e = f$ ). The first case yields ellipses and hyperbolas and their degenerations (straight lines or point). In the other case the conic section is a parabola.

## 3. Classification using a matrix $M$ [CS, 3.1-3]

By an operation  $H^T \cdot M \cdot H$ , where matrix  $H$  is either a rotation matrix

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or a translation matrix } T = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, \text{ matrix } M = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

can be transformed to  $\begin{pmatrix} a' & 0 & 0 \\ 0 & c' & 0 \\ 0 & 0 & f' \end{pmatrix}$ , and, in case  $a \cdot c - b^2 = 0$ , to  $\begin{pmatrix} a' & 0 & 0 \\ 0 & 0 & e' \\ 0 & e' & 0 \end{pmatrix}$  (or respec-

tively to  $\begin{pmatrix} 0 & 0 & d' \\ 0 & c' & 0 \\ d' & 0 & 0 \end{pmatrix}$ ). As the kind of a conic section is neither changed by a translation

nor by a rotation, we can use matrices for the classifications provided in paragraph 2.

For doing this we introduce new quantities:  $\Delta = \det(M)$ ,  $\delta = \det \begin{pmatrix} a & b \\ c & c \end{pmatrix}$  and  $S = a + c$ .

They are invariant with respect to the rotations and translations from above. So we get:

$\delta$	$\Delta \neq 0$	$\Delta = 0$
<b>&gt;0</b>	$\Delta \cdot S < 0$ : Ellipse	Point
	Empty set	
<b>&lt;0</b>	Hyperbola	Intersecting pair of straight lines
<b>=0</b>	Parabola	Pair of parallel straight lines

Table 3

© The general case first

$$m := \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} ; \text{rot} := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \text{tra} := \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$$

© the determinants

$$\{\det(m), \det(\text{rot}), \det(\text{tra})\} \quad \{a \cdot (c \cdot f - e^2) - b^2 \cdot f + 2 \cdot b \cdot d \cdot e - c \cdot d^2, 1, 1\}$$

© the translation (with u and v from 2.3)

$$m1 := \text{tra}^T \cdot m \cdot \text{tra} ; u = \frac{b \cdot e - c \cdot d}{a \cdot c - b^2} \text{ and } v = \frac{-(a \cdot e - b \cdot d)}{a \cdot c - b^2}$$

$$\begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & \frac{-(b \cdot e - c \cdot d)^2}{(a \cdot c - b^2) \cdot c} - \frac{e^2}{c} + f \end{bmatrix}$$

$\det(m1) = \det(m)$  true

© the rotation of m1 (with  $\theta$  from 2.4)

$$m2 := \text{rot}^T \cdot m1 \cdot \text{rot} ; \theta = \frac{1}{2} \cdot \tan^{-1} \left( \frac{2 \cdot b}{a - c} \right)$$

$$\begin{bmatrix} (a-c) \left( \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \right)^2 + 2 \sin \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cdot b + c & 2 \left( \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \right)^2 \cdot b + (c-a) \sin \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) - b & 0 \\ 2 \left( \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \right)^2 \cdot b + (c-a) \sin \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) - b & (c-a) \left( \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \right)^2 - 2 \sin \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cos \left( \frac{\tan^{-1} \left( \frac{2 \cdot b}{a-c} \right)}{2} \right) \cdot b + a & 0 \\ 0 & 0 & \frac{-(b \cdot e - c \cdot d)^2}{(a \cdot c - b^2) \cdot c} - \frac{e^2}{c} + f \end{bmatrix}$$

© this matrix must be simplified using tCollect and specifications for  $\text{sign}(a-c)$  and  $|a-c|$

$$m2 := \text{tCollect}(m2)$$

$$\begin{bmatrix} 2 \cdot \frac{1}{a^2 - 2 \cdot a \cdot c + 4 \cdot b^2 + c^2} \cdot \text{sign}(a-c) \cdot b^2 + \left( \frac{a-c}{2} \right) \cdot \frac{1}{a^2 - 2 \cdot a \cdot c + 4 \cdot b^2 + c^2} \cdot |a-c| + \frac{a+c}{2} & 0 & 0 \end{bmatrix}$$

16/99

Again a lot of algebra!

#### 4. Representation of Conic Sections

In this section some constructional ideas for conic sections are presented. All these ideas use the 'zeros' command from TI-Nspire CAS. (So they can not be used in the nonCAS version of TI-Nspire.)

If  $p(x,y)$  is a 2<sup>nd</sup> degree polynomial in  $x$  and  $y$  then  $\text{zeros}(p(x,y), y)$  yields a list of (root-) expressions in  $x$ . The G&G application interprets this list as a family of functions in  $x$  with two elements maximum. So a definition  $f1(x):=\text{zeros}(p(x,y),y)$  in G&G plots a graph with two colors and labels  $f1\_1$  and  $f1\_2$ .

Many thanks to Philippe Fortin (France) for providing most of the ideas realized in the following paragraphs.

#### 4.1 The Conic Section Generator [CS, 4]

The classification of conic sections from paragraph 2 and 3 can easily be studied with the ConicSectionGenerator (CSG).

The workspace is divided into two G&G windows, a (smaller) Geometry and a (wider) Graphs window. In the left window we insert 6 sliders for variables  $a$  through  $f$  having a range from -10 to 10, which can easily be changed. The step size is 0.1 for  $a, b, c$ , and 0.5 for  $d, e, f$  and may also be changed as needed.

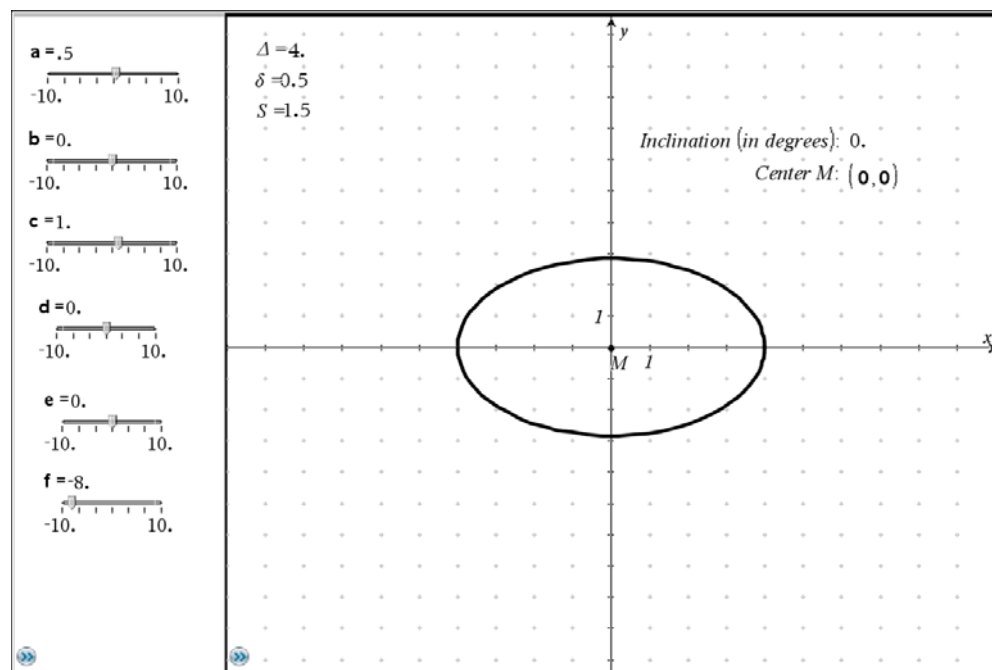
The right window has the definition

$f1(x):=\text{zeros}(a \cdot x^2 + 2b \cdot x \cdot y + c \cdot y^2 + 2d \cdot x + 2e \cdot y + f = 0, y)$ , (in function mode) which plots the (maximum) two components of the conic section.

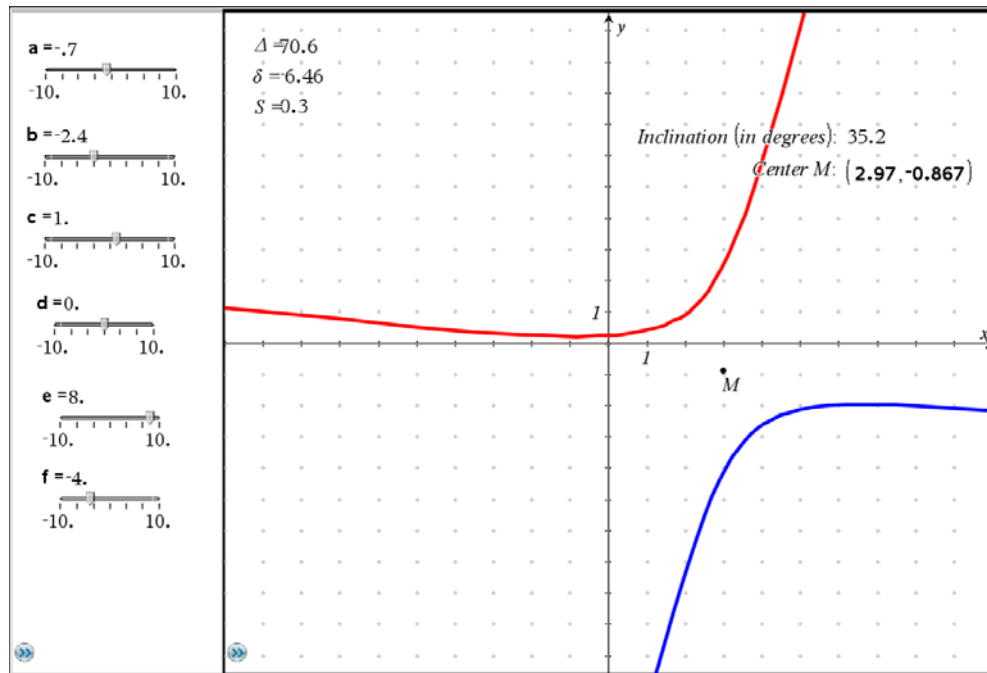
In addition the determinants  $\Delta = \det \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$ ,  $\delta = \det \begin{pmatrix} a & b \\ c & c \end{pmatrix}$  and the sum  $S = a + c$  are shown in the upper left corner.

In the upper right corner we see angle of inclination of the main axis and the coordinates of the centre point.

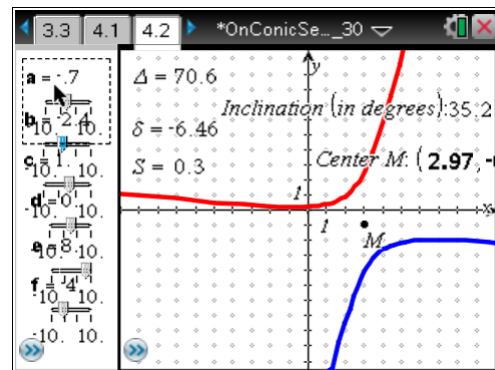
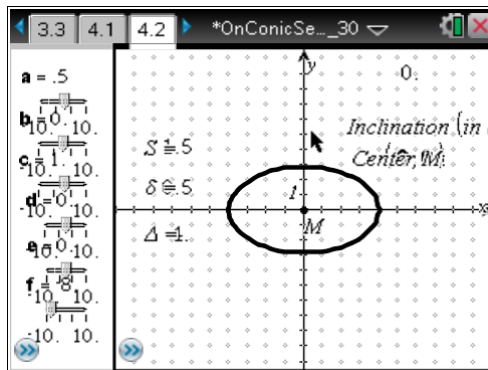
By modifying parameters  $a$  through  $f$  ellipses, hyperbolas, parabolas and straight lines can be created and can be compared with the summaries given in tables 1, 2 and 3.



This is the screen after moving some sliders, i.e. changing some coefficients:



Below are the screens from above as they are presented on the handheld Nspire:



#### 4.2 The Focus Feature [CS, 5]

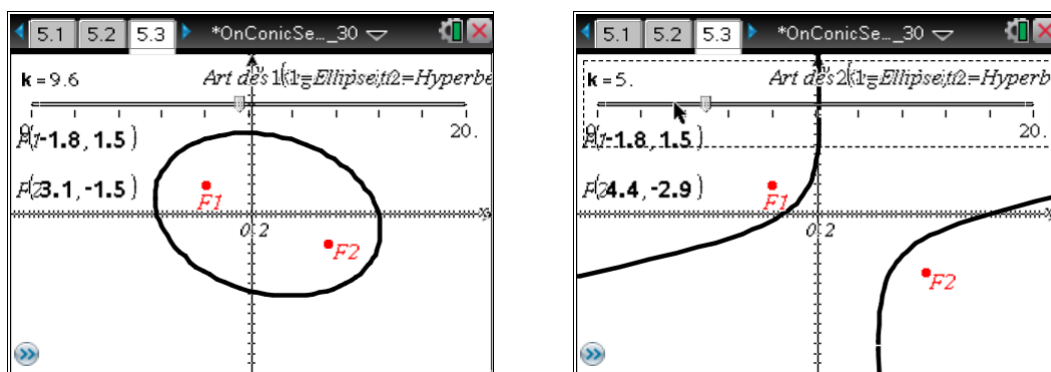
For ellipses the sum of the focus rays, i.e. the line segments connecting one point on the ellipse with the two focus points is constant. That means  $d(F_1X) + d(F_2X) = k$  ( $= \text{const.}$ ). For a hyperbola this condition is  $|d(F_1X) - d(F_2X)| = k$ . Both expressions on the left side are containing roots. One can eliminate these roots by quite elaborate algebraic transformations [see CS, 5.2] in order to obtain a rather complicated expression in  $x$  and  $y$  which depends on the focus point coordinates  $f_{11}, f_{12}, f_{21}, f_{22}$ , and the constant  $k$ .

The value of  $k$  can be varied by a slider. For the focus point coordinates we store the coordinates of two arbitrary points in  $f11, f12$  and  $f21, f22$  respectively. Now the graph can be plotted, using again the zeros function. By dragging the foci and changing  $k$  we can create ellipses and hyperbolas [see CS 5.3].

In general it would also be possible to insert the focus point condition directly in G&G, that means to define  $f1(x) = \text{zeros}(\text{abs}(\text{norm}([x;y]-[f11;f12])-\text{norm}([x;y]-[f21;f22]))-k,y)$ .

If previously the coordinates and  $k$  have been created like mentioned above we should get the graph of a conic section. However, it takes quite a long time (depending on the performance of the computer) to receive a result. And if the foci are dragged around or  $k$  is changed Nspire is blocked for several minutes and the dynamics of the application gets lost.

A final remark: As for this example there are not so many objects on the screen, it can also run on a TI-Nspire CAS Handheld device (with reasonable visibility). The complicated calculations from [CS, 5.2] are omitted and just the result is inserted. The file Focus\_NN.tns, where again NN=20 indicates a document for Nspire version 2.x and NN=30 is for version 3.x is among the files connected with this DNL.



Two handheld-screenshots of FOCUS

#### 4.3 The Five Point Feature [CS, 6]

The quadratic  $a \cdot x^2 + 2b \cdot x \cdot y + c \cdot y^2 + 2d \cdot x + 2e \cdot y + f = 0$  in  $x$  and  $y$  contains 6 coefficients. With the coordinates of five different points inserted for  $x$  and  $y$  we get five equations. An equation like  $a + 2b + c = 1$  avoids that  $a$ ,  $b$  and  $c$  are zero at a time and we call it normalization condition.

This system yields a unique solution for  $a$  through  $f^{(1)}$ .

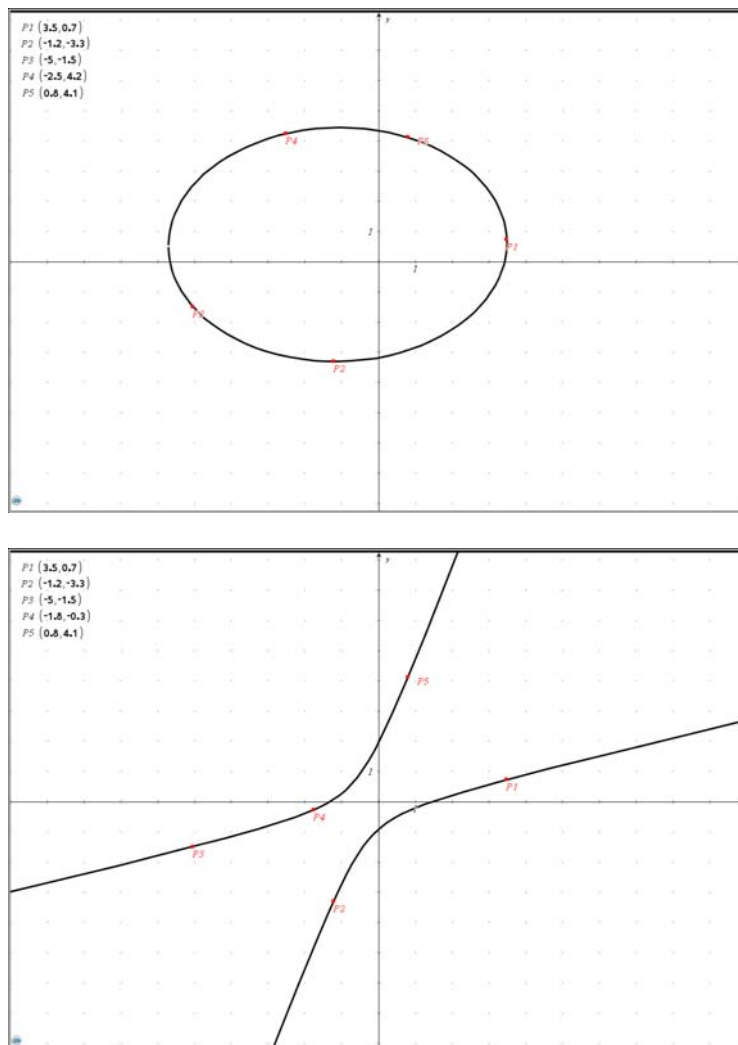
For the implementation we define a function `fivepoint(P1, P2, P3, P4, P5)` where  $P1, \dots, P5$  are 2-dimensional column vectors with components  $p1, \dots, p5$  and  $q1, \dots, q5$ . This function solves the system mentioned above and assigns the solution to

$$a \cdot x^2 + 2b \cdot x \cdot y + c \cdot y^2 + 2d \cdot x + 2e \cdot y + f.$$

We only have to draw five different points in a G&G window, assign their coordinates to  $p1, \dots, p5$  and  $q1, \dots, q5$  and define  $f1(x) := \text{zeros}(\text{fivepoint}(P1, P2, P3, P4, P5), y)$ . A conic section through those five points appears. Now we can grab one or another point and drag it around observing what is happening.

I have provided a stand alone document to run on handhelds. It is called `FivePoint_NN.tns` and consists of only two pages: a short introduction and the construction in G&G. The function `fivepoint` is contained in the document but no longer directly visible.

Two screenshots of `fivepoint` are given on the next page.



<sup>[1]</sup> Additional Comments of the Editor:

It might be a good question for students, how to explain the fact that there are 6 unknown coefficients in the quadratic but only five points are sufficient to define a conic?

Dividing (1) by  $f$  yields  $a'x^2 + b'xy + c'y^2 + d'x + e'y + 1 = 0$  with  $a' = \frac{a}{f}, b' = \frac{2b}{f}, \dots$ . Now it is easy to see that 5 points are sufficient. (One could also divide by  $a$ , ...).

TI-Nspire has implemented a long desired *DERIVE*-feature: Background pictures.

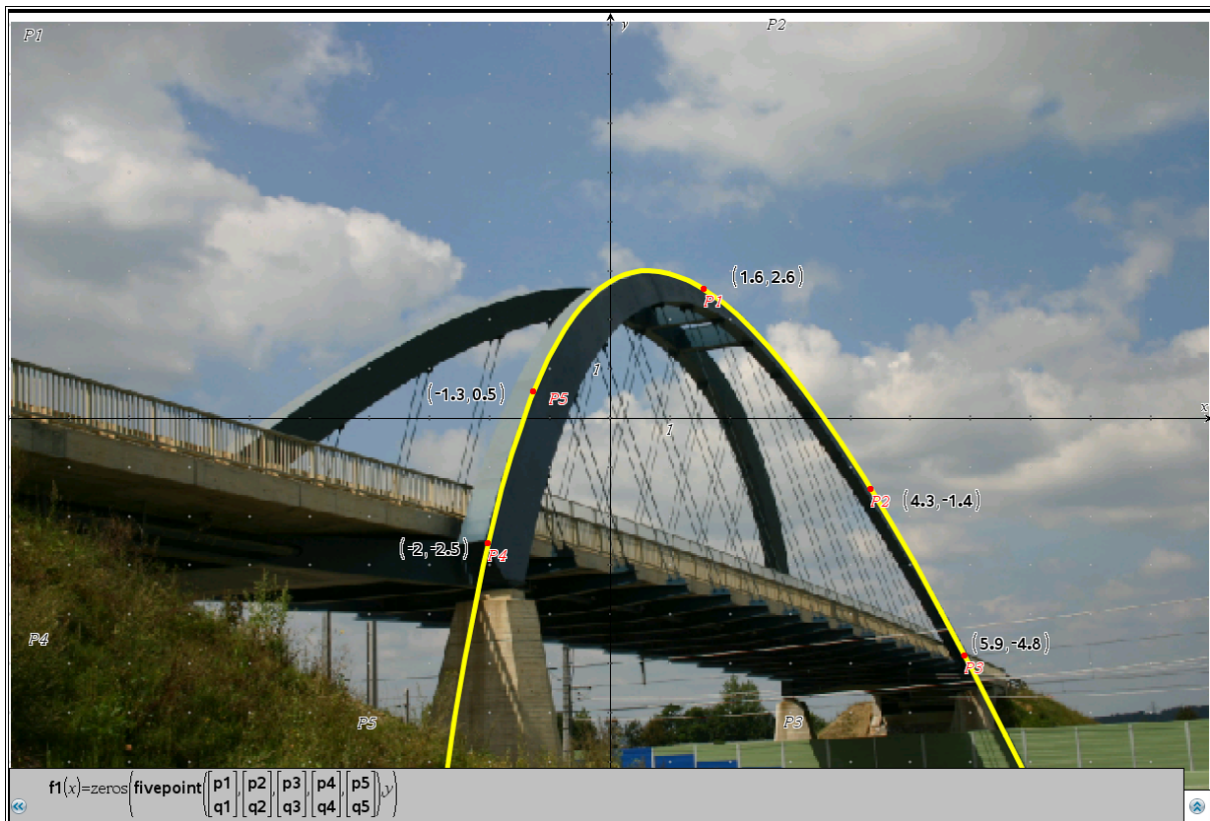
I load a picture of a bridge (jpg, bmp, ...) not far from my home and would like to analyze the form of the arc. Ask students (and teachers!!) and there first answer will be: This is a parabola.

Now drag the five points as accurate as possible on one of the arcs and you will find a wonderful curve. Go back to the Calculator App and ask for  $f_1(x)$ . As Wolfgang mentioned above you will find two branches, take one of them, do a little algebra and get the quadratic. Then follow the recipe given in CS for classifying and analyzing the curve. Or you write a program which does the job for you!

You could also use the CSG (page 10) to model the bridge (or other pictures). This is a little bit more sophisticated but the students might find out the impact of the various parameters.

Finally, there is a German version of this contribution at

<http://www.ti-unterrichtsmaterialien.net/imgserv.php?id=Pröpper.pdf>



Let's do a little algebra to find the quadratic in its implicit form:

$$\begin{aligned}
 f1(x) &= \left\{ -37.4498 \cdot \left( \sqrt{x^2 + 10.4995 \cdot x + 34.7117} - 0.910278 \cdot (x + 6.55427) \right), 37.4498 \cdot \left( \sqrt{x^2 + 10.4995 \cdot x + 34.7117} + 0.910278 \cdot (x + 6.55427) \right) \right\} \\
 y &= f1(x)[1] \\
 y &= -37.4498 \cdot \left( \sqrt{x^2 + 10.4995 \cdot x + 34.7117} - 0.910278 \cdot (x + 6.55427) \right) \\
 y &= -37.449809063834 \cdot \left( \sqrt{x^2 + 10.499493646781 \cdot x + 34.711708292528} - 0.91027781213277 \cdot (x + 6.5542685029883) \right) \\
 &\quad - 37.4498 \\
 &\quad - 0.026702 \cdot y = 1 \cdot \left( \sqrt{x^2 + 10.4995 \cdot x + 34.7117} - 0.910278 \cdot (x + 6.55427) \right) \\
 &\quad \left( -0.026702412295927 \cdot y = 1.0000002420262 \cdot \left( \sqrt{x^2 + 10.499493646781 \cdot x + 34.711708292528} - 0.91027781213277 \cdot (x + 6.5542685029883) \right) + 0.910278 \cdot (x + 6.55427) \right. \\
 &\quad \left. 0.910278 \cdot x - 0.026702 \cdot y + 5.96621 = 1 \cdot \left( \sqrt{x^2 + 10.4995 \cdot x + 34.7117} - 3.24439 \cdot x + 0.000001 \right) \right) \\
 &\quad \left( 0.910278 \cdot x - 0.026702412295927 \cdot y + 5.96620778706 = 1.0000002420262 \cdot \sqrt{x^2 + 10.499493646781 \cdot x + 34.711708292528} \right)^2 \\
 &\quad 0.828606 \cdot (x - 0.029334 \cdot (y - 223.433))^2 = 1 \cdot (x^2 + 10.4995 \cdot x + 34.7117) \\
 \text{expand}\left(\text{left}\left(0.828606037284 \cdot (x - 0.029334348732944 \cdot (y - 223.43328838384))^2 = 1.0000004840525 \cdot (x^2 + 10.499493646781 \cdot x + 34.711708292528)\right) - \text{right}\left(0.82860603 \cdot \right.\right. \\
 &\quad \left. \left. - 0.171394 \cdot x^2 - 0.048613 \cdot x \cdot y + 0.362317 \cdot x + 0.000713 \cdot y^2 - 0.318624 \cdot y + 0.88391 = 0 \right) \right) \\
 &= 0
 \end{aligned}$$

The next contribution will do this job for you!



# A Conics Explorer and a Conics Trainer

Josef Böhm, Würmla, Austria

As promised on page 13 here is the analysis of the “bridge – conic”.

$$\text{task} := -0.171394 \cdot x^2 - 0.048613 \cdot x \cdot y + 0.362317 \cdot x + 0.000713 \cdot y^2 - 0.318624 \cdot y + 0.88391 = 0$$

ans

Type	Center	Real axis 2a	Imaginary axis 2b	Real Vertices
Hyperbola	[-5.249788684, 44.47126414]	83.73679034	12.79439536	$\begin{bmatrix} -10.99451200 & 85.94367265 \\ 0.4949346356 & 2.998855624 \end{bmatrix}$
Asymptotes				
$\begin{bmatrix} 0.09253588869 \cdot t - 5.249788684 & 6.620111512 \cdot t + 44.47126414 \\ 1.888548332 \cdot t - 5.249788684 & 44.47126414 - 6.345693384 \cdot t \end{bmatrix}$				

It seems to be a hyperbola. I say “it seems to be” because unlike to translation and rotation the conics can change their type after applying a perspective projection which is given by taking a picture of the object with a camera. So it could be indeed a parabola!!

In fact, the program would not give the answer parabola - even if it would be one - because of numerical reasons. As Wolfgang explained on page 6 the discriminant must become zero – and I am quite sure that the pixel coordinates would not allow this.

This example was the introduction.

The **Conics Explorer** is the core part of my **Conics Trainer**:

Load `conics.mth` as a Utility file and simply Simplify start

```
#1: LOAD(D:\DOKUS\DNLS\DNL11\MTH83\conics.mth)
```

```
#2: start
```

**Conics Trainer – Trainer für Kegelschnitte**

**con1 = Center in origin unrotated – Kegelschnitt in Ursprungslage**

**con2 = shifted origin, unrotated – verschobener Kegelschnitt**

**con3 = conic shifted and rotated – Kegelschnitt in allgemeiner Lage**

**con = random choice from above – Zufallsaufgabe aus den obigen**

**You can enter your own conic as task:= ...**

**ans = gives the analysis; ant = liefert die Analyse**



Let's assume that your students should train the conics of type 2 (shifted origin and not rotated):

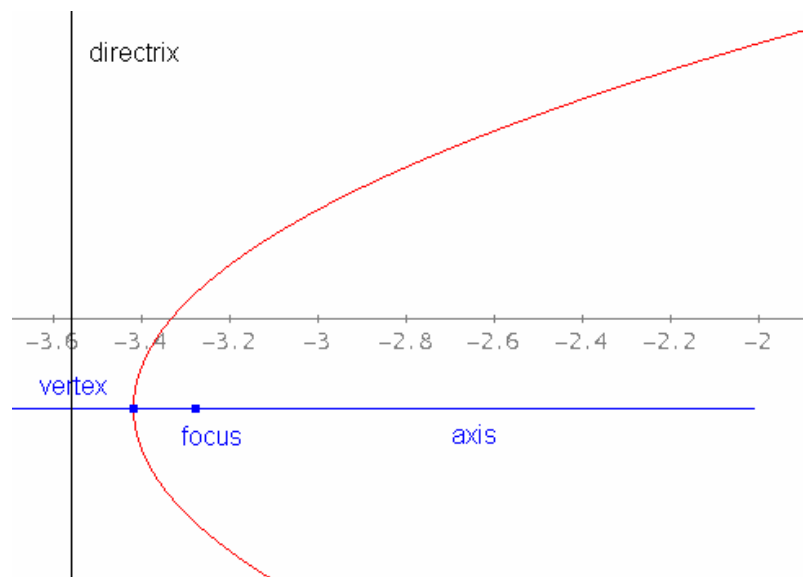
con2= delivers a randomly generated quadratic.

The students do their investigation with or without technology and then they can check their solution.

#4:  $\text{con2} = (9 \cdot x - 16 \cdot y)^2 - 7 \cdot y = -30$

#5:  $\text{ans} = \left[ \begin{array}{ccccc} \text{Type} & \text{Vertex} & \text{Parameter} & \text{Focus} & \text{Axis \& Directrix} \\ \text{Parabola} & \left[ -\frac{1969}{576}, -\frac{7}{32} \right] & \frac{9}{32} & \left[ -\frac{59}{18}, -\frac{7}{32} \right] & \left[ \begin{array}{cc} -\frac{9 \cdot t}{64} - \frac{1969}{576} & -\frac{7}{32} \\ -\frac{1025}{288} & \frac{9 \cdot t}{64} - \frac{7}{32} \end{array} \right] \end{array} \right]$

Finally they can plot the curve together with all the important details.



con3= gives the most complicated cases – shifted and rotated.

#6:  $\text{con3} = (12 \cdot x^2 + 9 \cdot x \cdot y + 8 \cdot x + 12 \cdot y^2 - 17 \cdot y = -16)$

#7: ans

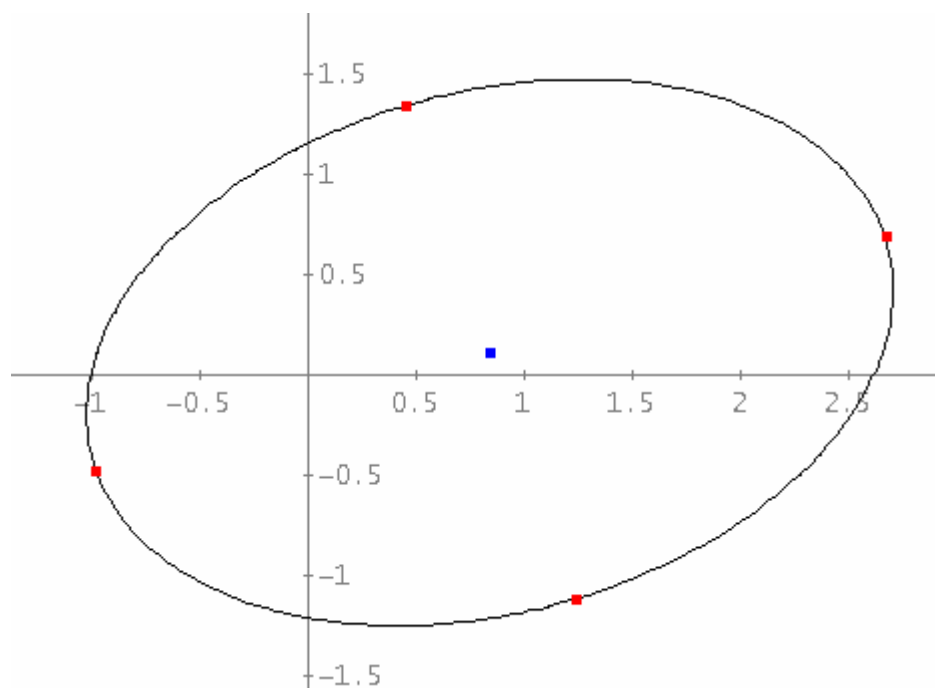
#8:  $\left[ \begin{array}{ccccc} \text{Typ} & \text{Mittelpkt} & \text{Hauptachse} & \text{Nebenachse} & \text{Scheitel} \\ \text{Type} & \text{Center} & \text{Major axis} & \text{Minor axis} & \text{Vertices} \\ \text{Ellipse} & [-0.6969696969, 0.9696969696] & \text{imaginary} & \text{imaginary} & \text{imaginary vertices} \end{array} \right]$

#9:  $\text{con3} = (8 \cdot x^2 - 5 \cdot x \cdot y - 13 \cdot x + 15 \cdot y^2 + y = 21)$

#10: ans

#11:  $\left[ \begin{array}{ccccc} \text{Typ} & \text{Mittelpkt} & \text{Hauptachse} & \text{Nebenachse} & \text{Scheitel} \\ \text{Type} & \text{Center} & \text{Major axis} & \text{Minor axis} & \text{Vertices} \\ \text{Ellipse} & [0.8461538461, 0.1076923076] & 3.833364128 & 2.587418828 & \left[ \begin{array}{cc} 2.671401567 & 0.6926204161 \\ -0.9790938750 & -0.4772358007 \\ 0.4513429306 & 1.339685972 \\ 1.240964761 & -1.124301356 \end{array} \right] \end{array} \right]$

The quadratic in expression #6 turns out as an imaginary conic, but in expression #9 we find a rotated and shifted ellipse.



The program code:

```

start_(dummy) :=
  Prog
    dummy := RANDOM(0)
    [x : Complex, y : Complex]
    DISPLAY("Conics Trainer - Trainer für Kegelschnitte")
    DISPLAY("con1 = Center in origin unrotated - Kegelschnitt in Ursprungs Lage")
#1:  DISPLAY("con2 = shifted origin, unrotated - verschobener Kegelschnitt")
    DISPLAY("con3 = conic shifted and rotated - Kegelschnitt in allgemeiner Lage")
    DISPLAY("con = random choice from above - Zufallsaufgabe aus den obigen")
    DISPLAY("")
    DISPLAY("ans = gives the analysis; ant = liefert die Analyse")
    DISPLAY("")

#2:  start := start_()

#3:  task :=

conic1(a, b, c, rd) :=
  Prog
    rd := RANDOM(8)
  Loop
    a := 20 - RANDOM(41)
    b := 20 - RANDOM(41)
#4:  f := 30 - RANDOM(61)
    If f·a·b ≠ 0
      Prog
        task := [a·x^2 + b·y^2 - f = 0, a·x^2 + b·y = 0, a·y^2 + b·x = 0]
        task := IF(rd < 6, task↓1, IF(rd < 7, task↓2, task↓3))
      RETURN task

```

```

conic2(a, b, c, d, e, rd) :=
  Prog
  rd := RANDOM(8)
  a := 20 - RANDOM(41)
  b := 20 - RANDOM(41)
  c := 20 - RANDOM(41)
  d := 20 - RANDOM(41)
  e := 30 - RANDOM(61)
#5: task := [a·x^2 + b·y^2 + c·x + d·y + e = 0, a·x^2 + c·x + d·y + e = 0, b·y^2 + c·x + d·y + e = 0]
  task := IF(rd < 6, task_1, IF(rd < 7, task_2, task_3))
  RETURN task

conic3(a, b, c, d, e, f) :=
  Prog
  a := 20 - RANDOM(41)
  b := 20 - RANDOM(41)
  c := 20 - RANDOM(41)
  d := 20 - RANDOM(41)
  e := 20 - RANDOM(61)
#6: f := 30 - RANDOM(61)
  task := a·x^2 + b·y^2 + c·x + d·y + e·x·y + f = 0
  RETURN EXPAND(task)

#7: con1 := conic1()
#8: con2 := conic2()
#9: con3 := conic3()
con := [con1, con2, con3]
#10: RANDOM(3) + 1

```

Expression #11 (not printed here) is the main function `ch_conic()`. It investigates the random conic `con` from #10 which is also stored as global variable `task` and stores the results in `ans` and `ant` (for the German users!).

```

#12: ans := ch_conic(task)
#13: ant := ch_conic(task)

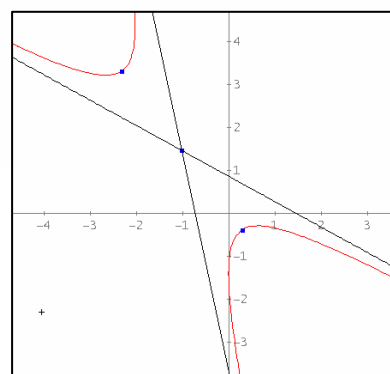
```

Let's try finding a hyperbola:

$$\text{con3} = (18 \cdot x^2 + 34 \cdot x \cdot y - 13 \cdot x + 6 \cdot y^2 + 17 \cdot y = -12)$$

ans

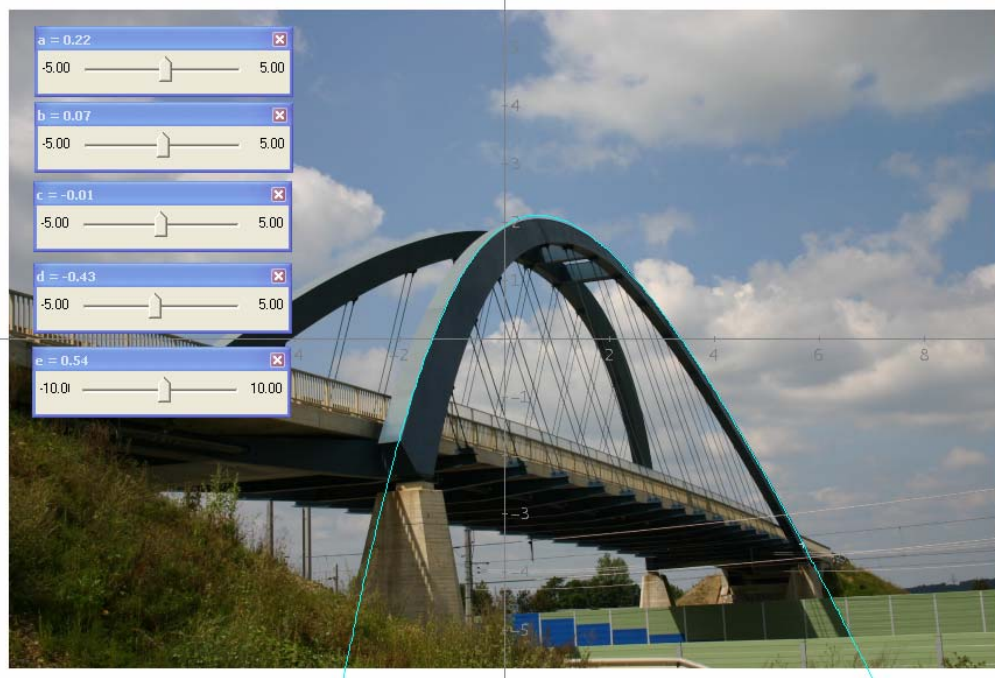
Type	Center	Real axis 2a	Imaginary axis 2b
Hyperbola	$[-1.013812154, 1.455801104]$	4.532953869	2.030942852
Real Vertices		Asymptotes	
$-2.322866802$	$3.306015585$	$-0.4727708474 \cdot t - 1.013812154$	$2.399597608 \cdot t$
$0.2952424933$	$-0.3944133754$	$2.105449700 \cdot t - 1.013812154$	$1.455801104 -$
$+ 1.455801104$			
$1.244452697 \cdot t$			



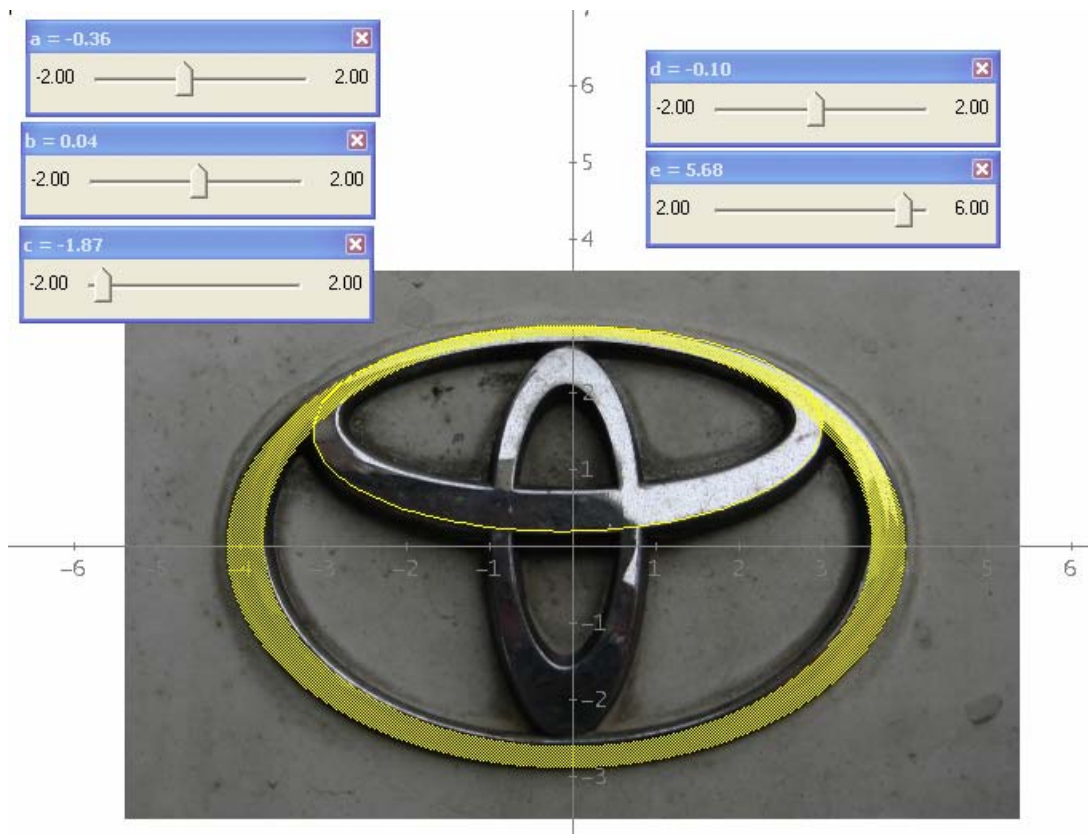
The next task is not so easy.

Model the arc as a conic using the slider bars:

$$\#1: \quad a \cdot x^2 + b \cdot x \cdot y + c \cdot y^2 + d \cdot x + e \cdot y = 1$$



You can find conics everywhere. I used the sliders to model the first three ellipses. Three more are remaining waiting to be plotted!



## Textbooks and the arctanh

## An email-discussion

**From:** Michel Beaudin

**To:** Albert Rich, David Jeffrey, David Stoutemyer, Josef Boehm

**Subject:** Textbooks and the arctanh

Hello Albert and my friends -- it is always exciting for me when I am writing an email to the "fathers of Derive", to my best ROC friend ("ROC" stands for Rest of Canada: Quebecers love this word) and to my best friend in Europe.

My question is: do you know why textbooks (at least calculus ones) are so quiet about the arctanh function? I understand that  $\operatorname{arctanh}(x)$  has the open interval  $-1 < x < 1$  as domain and this can cause a problem for students when they will be integrating  $1/(1 - x^2)$  if the bounds of their definite integral are, for example, 3 to 5. But it is quite easier for them to use the antiderivative  $\operatorname{arctanh}(x)$  instead of 2 logs (or one log with quotients) because of the symmetry with the antiderivative  $\operatorname{arctan}(x)$  for  $1/(1 + x^2)$ . Moreover, when textbooks will start to use  $\operatorname{arctanh}(x)$  for the antiderivative of  $1/(1-x^2)$ , it will be natural to perform the integral of  $\sec(x)$ :  $\sec(x) = 1/\cos(x) = \cos(x)/\cos^2(x) = \cos(x)/(1 - \sin^2(x))$ , so the substitution  $u = \sin(x)$  leads to  $1/(1-u^2)$  and that will be a good reason to use  $\operatorname{arctanh}(\sin(x))$  instead of  $\log(\operatorname{abs}(\sec(x) + \tan(x)))$  -- what textbooks are still using -- as an antiderivative of  $\sec(x)$ .

Looking back to an old copy of the DNL journal, David Stoutemyer wrote: "Calculus students are introduced to complex numbers in high-school algebra, and they are taught that antiderivatives can differ by arbitrary constants. Why not exploit logarithms of negative numbers to reinforce both ideas? This is not to say that we should generalize all elementary calculus to complex numbers -- only that if a complex number is most correct and natural in a few places, exploit it rather than fight it." (DNL #26, page 5).

Things do not change as fast as I would like!

Regards,  
Michel

Von: Albert Rich

An: Michel.Beaudin, David Jeffrey, David Stoutemyer, Josef Boehm

Betreff: Re: Textbooks and the arctanh

Hello Michel,

Not being an educator, I am probably least qualified of your addressees to know "why textbooks are so quiet about the arctanh function?" Or, more generally, why the inverse trigonometric functions get so much more "ink" than the inverse hyperbolic ones? This is despite the undeniable symmetry between these two sets of functions. This symmetry is evidenced by the identical, parallel treatments they receive in Chapter 4 of the recently released NIST Handbook of Mathematical Functions, which is the authoritative successor to Abramowitz and Stegun.

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So good luck on your quest to make the inverse hyperbolic functions (and hyperbolic functions?) first class objects of study.

The father of Rubi,

Albert

De : David Jeffrey

Objet : Re: Textbooks and the arctanh

My guess is that trigonometric functions are introduced early in high school because of the geometric application.

Hyperbolic functions, on the other hand, have to wait until the exponential function has been covered, and thereafter they never catch up to trig functions in the mathematical mind.

One advantage of  $\operatorname{arctanh}(\sin x)$  - or  $\operatorname{invtanh}(\sin x)$  as I prefer to write it - is that it is always a real value, unlike  $\ln(\sec x + \tan x)$ .

For the same reason  $\operatorname{arctanh}\left(\frac{x^2-1}{x^2+1}\right)$  is a better integral of  $1/x$  than  $\ln(x)$  but I don't think it will catch on.

DJJ

From: Michel Beaudin

Subject: RE: Textbooks and the arctanh

David, if the first goal when choosing an antiderivative is to have a real value, then I "prefer"  $\ln(\operatorname{abs}(x))$  instead of  $\operatorname{arctanh}\left(\frac{x^2-1}{x^2+1}\right)$  for the integral of  $1/x$  (these 2 expressions are identical). In the same idea,  $\ln(\operatorname{abs}(\sec x + \tan x))$  is always real, so those who are writing calculus textbooks will continue to use it instead of  $\operatorname{arctanh}(\sin x)$  for the integral of  $\sec x$ .

Michel

Von: David R Stoutemyer

Betreff: Re: RE: Textbooks and the arctanh

Hi Michel,

I can remain quiet no longer.

The absolute value:

1. unnecessarily makes the antiderivative incorrect for non-real endpoints (and some people will eventually use non-real endpoints not realizing the incorrectness),

2. completely hides the non-integrable singularity if someone integrates from a negative to a positive value. (With  $\ln(x)$  you get a some warning from the  $i\pi$  term.)

3. makes it very difficult to simplify subsequent expressions that use the antiderivative -- for examples, mixtures of  $\ln(x)$  and  $\ln(\text{abs}(x))$ . As another example, iterated integrals become very difficult if an inner integral generates a  $\ln(\text{abs}(x))$ .

With all of its extra closure, the complex domain is so much easier than the real domain. I don't know why we put these extra hurdle constraints on students who have been exposed to complex numbers in earlier courses.

-- aloha, david s.

Von: Michel.Beaudin

Betreff:\* RE: RE: Textbooks and the arctanh

Hi David,

You are absolutely right and I will add more!

After their calculus course, my students will take the ODE course. They will learn how to solve by hand "easy" ODEs and also they will learn how to use the "desolve" command of their CAS calculator (Voyage 200 and, starting in September, Nspire CX CAS). Those who have set the "complex format" to "Real" will sometimes get a "Non-real result" message on their screen when trying to solve some ODE. With complex format set to "Rectangular", we get a solution involving the number "i" but, choosing the appropriate domain -- and this is a nice opportunity to recall the existence and uniqueness of solution --, the "i" will "disappear" and students will obtain a real solution!

In my talk at ACA 2007 ("Using  $\ln(\text{abs}(x))$  as an Antiderivative for  $1/x$  is a Bad Choice!"), I used the separable ODE  $y' = 4*y/(x^2-9)$ ,  $y(0) = 5$ . In order to remove the imaginary unit "i" from the solution found by the device (and also Derive, Maple, ...), students have to use the fact that we are looking for a solution defined in the strip  $-3 < x < 3$  because the initial condition is at the point (0, 5) : according to the existence and uniqueness theorem, this ODE has a unique solution defined for  $-3 < x < 3$  because both  $f$  and  $\text{dif}(f, y)$  are continuous in some rectangle centred at the point (0, 5) with  $-3 < x < 3$  (here  $f$  stands for  $4*y/(x^2 - 9)$ ). And restricting the domain gives this unique and real solution!

Consequently, we are doing more and not less mathematics if we avoid absolute values in the integral of  $1/x$ .

Best regards,

Michel

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On 11-05-17 03:00 PM, Josef Böhm wrote:

Dear friends,

I am very fascinated by your high level discussion. Unfortunately I can not contribute because I never had to do with hyperbolic functions and their inverse. BTW, in German the inverse hyperbolic functions are called “Area-functions”, eg. Area sinus hyperbolicus, etc. I never taught on a technical school where the hyperbolic functions might be part of the curriculum.

What I can contribute is the fact that in Austria secans and cosecans are widely unknown and unused. I am quite sure if I’d ask some math teachers about secans and cosecans – they would have never heard about them. They don’t appear in our textbooks. I never heard about sec and cosec during my math study on the Technical University ...

Many thanks and I will follow your discussion with high interest – and I will not hesitate including it into the next DERIVE newsletter. It is a nice coincidence that I am just revising DNL#26 with David’s paper “ $\ln x$  or  $\ln |x|$ ?”

Best regards to you all,

Josef

From: David Jeffrey

Subject: Re: AW: RE: Textbooks and the arctanh

Dear All,

My Russian wife has been teaching North American calculus for the first time, and also commented that she was not taught secant and cosecant. I guess in Europe you are taught to write  $1 + \tan^2 = 1/\cos^2$  instead of our  $1 + \tan^2 = \sec^2$ .

We always boast that mathematics is a universal language, but maybe not quite as universal as we hope.

Some years ago, James Davenport was working on the open math project and reminded us of even more obscure functions such as  $\text{versin}(x) = 1 - \cos(x)$  See

<http://en.wikipedia.org/wiki/File:Circle-trig6.svg>

for an attractive diagram.

David J



Von: David R Stoutemyer

Betreff: Re: AW: RE: Textbooks and the arctanh

Tables of csc, sec and cot usefully replaced a division with a multiplication in the days of manual computation. However, those days are long gone, and most students quickly forget even whether csc is the reciprocal of cos or sin. Therefore we should stop wasting valuable curriculum time torturing students with these secondary trigonometric functions and their hyperbolic counterparts, just as we no longer torture students with other trivially related trigonometric functions such the haversine, which was convenient for manual celestial navigation computations. We don't have a special name for the reciprocal of  $\ln(\dots)$ ,  $\text{abs}(\dots)$ , etc.

Also, I think that hyperbolic functions are way less important than trigonometric, because exponentials are often a reasonable alternative to hyperbolics, whereas complex exponentials are a less attractive alternative to sinusoids. (Yes, even I prefer results absent  $i$  when  $i$  offers no strong advantage.)

-- aloha, david s.

Von: Michel.Beaudin@etsmtl.ca

Gesendet: Mittwoch, 18. Mai 2011 02:22

An: nojo.boehm@pgv.at

Betreff: RE : RE: Textbooks and the arctanh

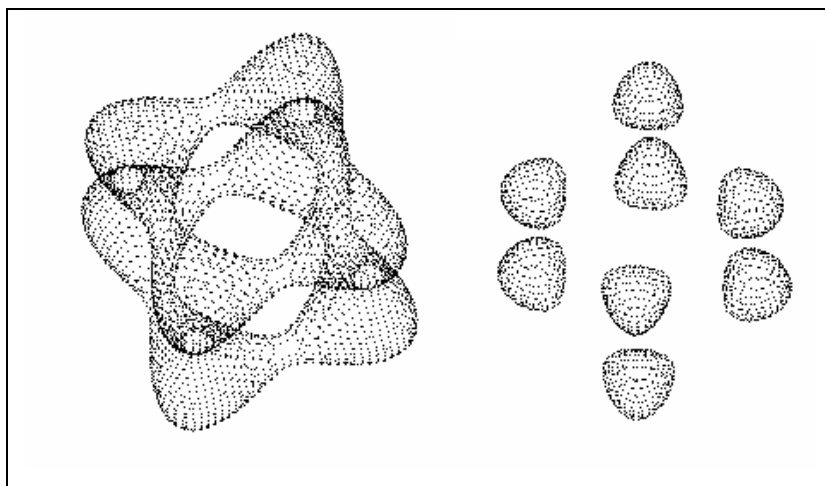
Dear Josef, this email is sent only to you because, some months ago, I sent my "new integral table" to David Jeffrey and Albert Rich but not to you (David Stoutemyer also received it). In fact, this "high level discussion" started last November when I was teaching single variable calculus (see the abstract of my Derive session) and when I started to be fascinated by Rubi! So, here is the table of integrals that I would like to use and also here is the table actually used at ETS in the single variable calculus course. If you can't make it to Houston, this table of integrals will give you an idea of what I will be talking about in my lecture!

Best regards,

Michel

Do you know this solid?

Find more on page 38!



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This is Michel Beaudin's very individual table of integrals (which he is finding to be sufficient):

1. $\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n+1 \neq 0)$	2. $\int \frac{1}{x} dx = \ln x$
3. $\int e^x dx = e^x$	
4. $\int \sin x dx = -\cos x$	5. $\int \cos x dx = \sin x$
6. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$	7. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \operatorname{arctanh}\left(\frac{x}{a}\right)$

8. $\int \tan x dx = -\ln(\cos x)$	9. $\int \cot x dx = \ln(\sin x)$
10. $\int \sec x dx = \operatorname{arctanh}(\sin x)$	11. $\int \csc x dx = -\operatorname{arctanh}(\cos x)$
12. $\int \sec^2 x dx = \tan x$	13. $\int \csc^2 x dx = -\cot x$
14. $\int \sec x \tan x dx = \sec x$	15. $\int \csc x \cot x dx = -\csc x$

16. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right)$
17. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \operatorname{arcsinh}\left(\frac{x}{a}\right)$
18. $\int \frac{1}{\sqrt{-a^2 + x^2}} dx = \operatorname{arctanh}\left(\frac{\sqrt{-a^2 + x^2}}{x}\right)$
19. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \int \frac{1}{\sqrt{a^2 - x^2}} dx$
20. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \int \frac{1}{\sqrt{a^2 + x^2}} dx$
21. $\int \sqrt{-a^2 + x^2} dx = \frac{x}{2} \sqrt{-a^2 + x^2} - \frac{a^2}{2} \int \frac{1}{\sqrt{-a^2 + x^2}} dx$

An email came in, which led to an interesting application (translation of the most important parts of the mail next page).

Sehr geehrter Herr Böhm,

durch Zufall bin ich auf Ihre e-mail Adresse gestoßen und möchte Sie, wenn es für Sie keine allzu große Mühe macht, um eine mathematische Hilfe bitten. Ich habe gehört, dass die Lösung meines Problems mit DERIVE möglich wäre, doch kann ich selbst damit nicht umgehen.

Einige Sätze zu meiner Person: Ich bin 1939 geboren, habe in München an der TU Maschinenbau studiert, war dann mein ganzes Berufsleben in einer Turbinenfabrik in Nürnberg beschäftigt. In relativ jungen Jahren wurde ich zum Leiter der technischen Berechnung und der technischen EDV-Abteilung gemacht und hatte mit Thermodynamik, Strömungsmechanik, Festigkeitslehre, Schwingungslehre, Materialfragen, Lagerauslegung, Wellenschwingungen und Wuchttechnik zu tun. Mein ganzes Berufsleben hat mir viel Spaß gemacht. Als "technisches Gewissen" unserer Vertriebsleute hatte ich die Möglichkeit fast die ganze Welt zu bereisen doch gesehen hatte ich dabei relativ wenig weil nie Zeit war.

Nun im Ruhestand denke ich manchmal über Probleme nach die mich während meines Berufslebens irgendwie begleitet haben, doch deren Lösung aus Zeitmangel immer verschoben wurde.

Es geht um das Wissensgebiet "konforme Abbildungen". Wenn Sie sich auch damit auskennen, dann überlesen Sie bitte meine nachfolgenden kurzen einführenden Worte.

Mittels konformer Abbildung kann man einen Kreis in eine tragflügelförmige Figur verwandeln. Diese Figur sieht dann auch so ähnlich aus wie ein Schaufelprofil. Meine Idee ist es nun ein gegebenes Schaufelprofil zurück zu transformieren und zu sehen, ob die Rücktransformation eine "schöne, ebenmäßige" Figur ergibt oder so etwas wie eine "Kartoffel". In diesem Fall würde ich daraus schließen, dass dieses vorgegebene Profil strömungstechnisch nicht optimal ist.

Gegeben sind die Koordinatenwerte eines Kreises mit vielen  $x = \dots$  und  $y = \dots$  als Funktion von Radius und den Mittelpunktskoordinaten.

Mit den Formeln  $u = x * (1 + k/(x^2 + y^2))$  und  $v = y * (1 - k/(x^2 + y^2))$  ergibt sich als transformierte Figur eine Schaufelprofil in einem Koordinatensystem  $u, v$ .

Mein Problem liegt nun darin, aus diesen beiden vorgenannten Formeln  $u = \dots$  und  $v = \dots$  die Werte  $x$  und  $y$  als Funktion von  $u$  und  $v$  darzustellen. Ich habe mir schon die Finger krumm gerechnet und bin zu keiner Lösung gekommen.

Man sagte mir allerdings, dass eine solche Formelumkehr mit DERIVE möglich wäre.

Da Sie sich offensichtlich mit diesem System auskennen, wage ich es, Sie hier um eine fachliche Hilfe zu bitten.

Ich würde mich sehr über eine Antwort freuen, gegebenenfalls auch auf einen Hinweis wie ich hier weiterkommen könnte.

Viele Grüße aus Nürnberg

Klaus Körner

## Klaus Körner's problem – Inverse Conformal mappings

... My problem deals with the field of knowledge "conformal mappings".

Applying a conformal mapping it is possible to transform a circle into a figure similar to a wing of an airplane. This figure is also looking similar to a coop profile. My idea is to transform a given coop profile back in order to check if the back transformation results in a "beautiful regular" figure or something like a "potato": in the latter case I would conclude that the given profile is not optimal in fluidic regard, it could be a "bluff body".

Given is a circle by its radius and the coordinates of its centre.

Applying the formulae  $u = x * (1 + k/(x^2 + y^2))$  and  $v = y * (1 - k/(x^2 + y^2))$  we obtain as transformed figure the profile of a wing in a uv-coordinate system.

My problem is to express the variables  $x$  and  $y$  as functions of  $u$  and  $v$ . I have calculated my "fingers awry" but I could not get a solution.

I was told that DERIVE should be able to solve the problem.

It would be great if you could provide any advice or the solution of the problem.

My (Josef's) answer was:

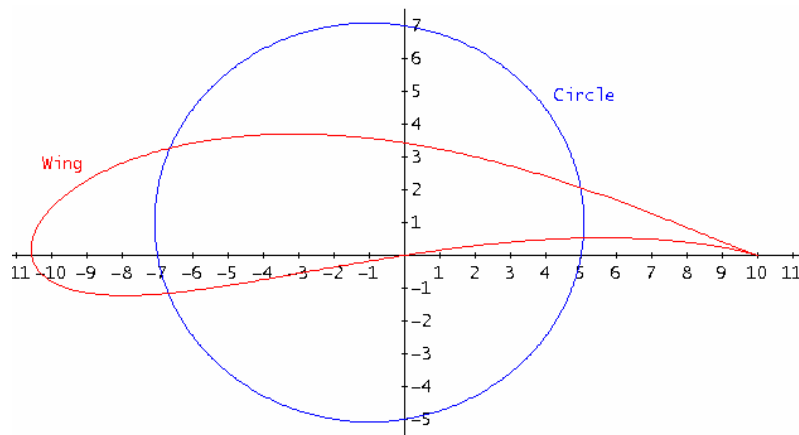
First of all I'd like to present the airplane wing profile as result of the conformal mapping given in Mr Körner's mail.

$$\#1: [\sqrt{37} \cdot \cos(t) - 1, \sqrt{37} \cdot \sin(t) + 1]$$

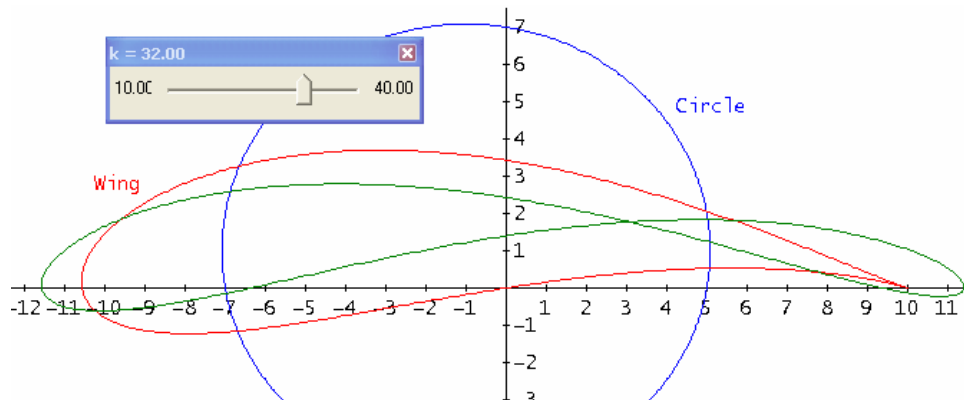
$$\#2: [x_- := \sqrt{37} \cdot \cos(t) - 1, y_- := \sqrt{37} \cdot \sin(t) + 1]$$

$$\#3: \text{SUBST} \left[ \left[ x \cdot \left( 1 + \frac{25}{x^2 + y^2} \right), y \cdot \left( 1 - \frac{25}{x^2 + y^2} \right) \right], [x, y], [x_-, y_-] \right]$$

$$\#4: \left[ \frac{2 \cdot (37 \cdot \cos(t))^2 - \sqrt{37} \cdot \cos(t) \cdot (\sqrt{37} \cdot \sin(t) + 33) + \sqrt{37} \cdot \sin(t) + 32}{2 \cdot \sqrt{37} \cdot \cos(t) - 2 \cdot \sqrt{37} \cdot \sin(t) - 39}, \right. \\ \left. \frac{(2 \cdot \sqrt{37} \cdot \sin(t) + 2) \cdot (\sqrt{37} \cdot \cos(t) - \sqrt{37} \cdot \sin(t) - 7)}{2 \cdot \sqrt{37} \cdot \cos(t) - 2 \cdot \sqrt{37} \cdot \sin(t) - 39} \right]$$



I generalized the mapping and introduced a slider for  $k$ . It was interesting to observe that only for  $k = 25$  a wing profile appeared.



Next step was dealing with Mr Körner's problem, i.e. solving the system of nonlinear equations for  $x$  and  $y$ . I assumed that DERIVE – with Groebner bases implemented – should be able to do the job.

#5: 
$$\text{SOLUTIONS} \left( u = x \cdot \left( 1 + \frac{25}{x^2 + y^2} \right) \wedge v = y \cdot \left( 1 - \frac{25}{x^2 + y^2} \right), [x, y] \right)$$

#6:

This did not work because the Groebner algorithm can be applied on systems of polynomials only. So I rewrote both equations free of fractions and then I tried again.

I kept the system as general as possible and replaced 25 by a variable  $k$ :

#7: 
$$\left( u = x \cdot \left( 1 + \frac{k}{x^2 + y^2} \right) \right) \cdot (x^2 + y^2) = (u \cdot (x^2 + y^2) = x \cdot (x^2 + y^2 + k))$$

#8: 
$$\left( v = y \cdot \left( 1 - \frac{k}{x^2 + y^2} \right) \right) \cdot (x^2 + y^2) = (v \cdot (x^2 + y^2) = y \cdot (x^2 + y^2 - k))$$

#9: 
$$\text{SOLUTIONS}(u \cdot (x^2 + y^2) = x \cdot (x^2 + y^2 + k) \wedge v \cdot (x^2 + y^2) = y \cdot (x^2 + y^2 - k), [x, y])$$

And I received a solution, not only one, but five including the trivial solution ( $x = y = 0$ ). But the others were bulky enough with two of them containing complex expressions.

$$\begin{aligned} & 0 \\ & \frac{\sqrt{2} \cdot ((\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 4 \cdot k \cdot u - v) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 8 \cdot u \cdot v)}{8 \cdot u \cdot v} \\ & \frac{\sqrt{2} \cdot (2 \cdot \sqrt{2} \cdot u \cdot v - (\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 4 \cdot k \cdot u - v) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 8 \cdot u \cdot v)}{8 \cdot u \cdot v} \\ & \frac{u}{2} - \frac{\sqrt{2} \cdot i \cdot (\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} + 4 \cdot k \cdot u - v) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 8 \cdot u \cdot v}{8 \cdot u \cdot v} \\ & \frac{u}{2} + \frac{\sqrt{2} \cdot i \cdot (\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} + 4 \cdot k \cdot u - v) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^2 + 2 \cdot u \cdot v + v^2)} - 8 \cdot u \cdot v}{8 \cdot u \cdot v} \end{aligned}$$

<b>p 28</b>	<b>Klaus Körner: Airplane and Conformal Mapping</b>	<b>D-N-L#83</b>
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I sent my results to Mr Körner and he – he does not use DERIVE – transferred the expressions to MS-Excel – and then had some satisfying results. He wrote (the most important parts of his letter will follow in a translation):

Sehr geehrter Herr Böhm,

nun habe ich endlich Zeit gefunden um Ihnen nochmals Dank zu sagen für die übermittelten Formeln als Ergebnis der DERIVE-Rechnungen.

Zunächst zu Ihrer Frage woher ich Ihre Adresse bekommen habe:

Nachdem ich vor mehreren Jahren von einem Geschäftsfreund in Südafrika auf DERIVE hingewiesen wurde (er hatte eine höhere Position in einer Raffinerie und war nebenbei „Freizeitmathematiker“) ist mir die Idee zur Nutzung dieses Programms für mein anstehendes Problem in den Sinn gekommen. Im Internet habe ich dann bei der „DERIVE USER GROUP“ Ihren Namen gefunden.

Im Internet habe ich dann auch (irgendwo) Ihren Artikel über die moderne Unterrichtung des Faches Mathematik gelesen und finde ihn echt zukunftsweisend (leider finde ich ihn nicht mehr.... Bitte um Hilfe).

Zu meiner Zeit und wahrscheinlich auch noch zu Ihrer Zeit wurde Mathematik sehr trocken gelehrt und dadurch hatte dieses Fach auch wenig Freude für die Schüler. Ich erinnere mich noch an meine Oberstufe wo Kurvendiskussion besprochen, doch niemals ein Beispiel aus dem täglichen Leben angefügt wurde. Dafür wurden wir mit sphärischer Trigonometrie (Himmelskunde) gequält.

Beim Studium (allgem. Maschinenbau an der TH München) waren die Mathe-Vorlesungen auch sehr schwer und oft undurchsichtig. Ein ganzes Semester beschäftigte sich mit Funktionentheorie, konformen Abbildungen (auch 3-dimensional) und auch mit geschlossenen und offenen Torusformen. Im Nachhinein halte ich das für ein Maschinenbaustudium für völlig übertrieben!

Doch nun zu unserem Rechenbeispiel:

Ich habe mir die Beschreibung des DERIVE5 – Programms von Frau Maria Koth, Universität Wien, angesehen und festgestellt welche traumhaften Möglichkeiten dieses Programm bietet.

...

Aus Ihren Formeln habe ich die 3. Lösung und die 4. Lösung programmiert und ganz eigenartigerweise festgestellt, dass teilweise die Ergebnisse der 3. Lösung richtig sind und teilweise die der 4. Lösung.

Damit Sie eventuell mein Beispiel mit den Zeichenmethoden von DERIVE nachempfinden können möchte ich folgende Daten nennen:

Der Kreis hat den Mittelpunkt  $M(-1;+1)$  und einen Radius von Wurzel aus 37. Damit kann man schrittweise (ich habe immer  $5^\circ$  genommen) die Umfangskoordinaten berechnen.

Mit den Ihnen bekannten Formeln  $u = \dots$  und  $v = \dots$  ergibt sich dann eine Form, die einem Tragflügel ähnelt.

Die berechneten u- und v-Werte habe ich nun verwendet, um mit den langen Gleichungen der 3. und 4. Lösung eine „Rück“-Transformation in Richtung Kreis zu erreichen.

Eigenartigerweise geht das dann nicht ganz ohne Probleme. Ich habe nämlich festgestellt, dass es am Vorzeichen des jeweiligen y-Werts des Ausgangskreises liegt. Im Bereich positiver y-Werte stimmt das Ergebnis der 4. Lösung, sonst das Ergebnis der 3. Lösung. Beim Vorzeichenwechsel von y machen die Ergebnisse der 3. und der 4. Lösung einen Sprung!

Da ich aber eine bestimmte Profilform vorgeben möchte um dann mit der Rücktransformation zu sehen welche Form herauskommt, kann ich ja nicht vorausahnen, ob y einen Vorzeichenwechsel macht oder nicht!

Ich hoffe doch und glaube auch, dass ich bei der Umsetzung dieser komplizierten Formeln nach EXCEL keinen Fehler gemacht habe!?!

Bitte probieren Sie es vielleicht einmal mit DERIVE.

Es gibt ja mehrere Versionen von DERIVE. Ich habe auch gehört, dass die Version DERIVE5 besser sein soll als DERIVE6.

Mit welcher Fassung arbeiten Sie?

Sollten Sie mit den Derive-Plot Funktionen auch die eigenartigen Ergebnisse wie ich mit meinen EXCEL-Methoden erhalten, dann bitte ich Sie doch mit Mathematica nochmals einen Versuch zu starten. Vielleicht kommt dann irgendeine andere Lösung heraus.

Als Anhang schicke ich Ihnen meine EXCEL-Ergebnisse. Darin können Sie ändern oder erweitern wie Sie möchten.

Für heute viele Grüße und nochmals „Danke“.

Klaus Körner

Ein Teil der Excel-Datei und die zugehörigen Grafiken folgen der englischen Übersetzung der wichtigen Teile des Briefes.

*... finally I found time to thank once more for the sent formulae which resulted from the DERIVE calculations.*

*...*

*Using your expressions I took the real solutions (#3 and #4) and found out that partially results based on #3 and partially based on #4 are correct.*

*If you would like to reproduce my example with DERIVE I'll give the following data:*

*The circle is given by its centre  $M(-1;+1)$  and its radius  $\sqrt{37}$ . I calculated the points of the circumference in  $5^\circ$ -steps. Then using the formulae  $u = \dots$  and  $v = \dots$  one obtains the figure which is similar to an airplane wing.*

*These u- and v-values were used applying the extended expressions of solutions #3 and #4 for achieving a „back transformation“ to get the circle.*

*Curiously this does not work without problems. I found out that the problems are connected with the sign of the respective y-values of the initial circle. For  $y > 0$  one has to take the 4<sup>th</sup> solution otherwise the 3<sup>rd</sup> one. When y changes its sign then the results of #3 and #4 show a jump.*

As I want to start with a certain given wing profile in order to investigate the form resulting from the back transformation I cannot guess in advance whether  $y$  changes its sign or not.

I hope – and I am pretty sure – that I didn't make a mistake by transferring the bulky expressions to EXCEL!?

Please try with DERIVE. There are several versions of DERIVE. I've heard that DERIVE 5 should be better than version 6. Which version do you use?

...

I attach my Excel-results. You are free to change or to extend the file as you like. Many thanks again.

Klaus Körner

### Parts of Klaus Körner's Excel worksheet

Calculations of conformal mapping and back transformation										
Centre of the circle		$x_m =$	-1	= Entry						
		$y_m =$	1	= Entry						
Squared radius of the circle		$r^2 =$	37	= Entry						
Factor		$k =$	25	= Entry						
Deg	Pts on Circumference		Conformal Mapping		Backtransformation 4th DERIVE Solution		Backtransformation 3rd DERIVE Solution		Combination of 4th and 3rd solution manually combined	
	x	y	u	v	x-new	y-new	x-new	y-new		
0	5,083	1,000	9,818	0,068	5,083	1,000	4,735	-0,932	5,083	1,000
5	5,060	1,530	9,587	0,161	5,060	1,530	4,527	-1,369	5,060	1,530
10	4,990	2,056	9,273	0,292	4,990	2,056	4,283	-1,765	4,990	2,056
15	4,875	2,574	8,885	0,457	4,875	2,574	4,010	-2,117	4,875	2,574
20	4,716	3,080	8,432	0,653	4,716	3,080	3,716	-2,427	4,716	3,080

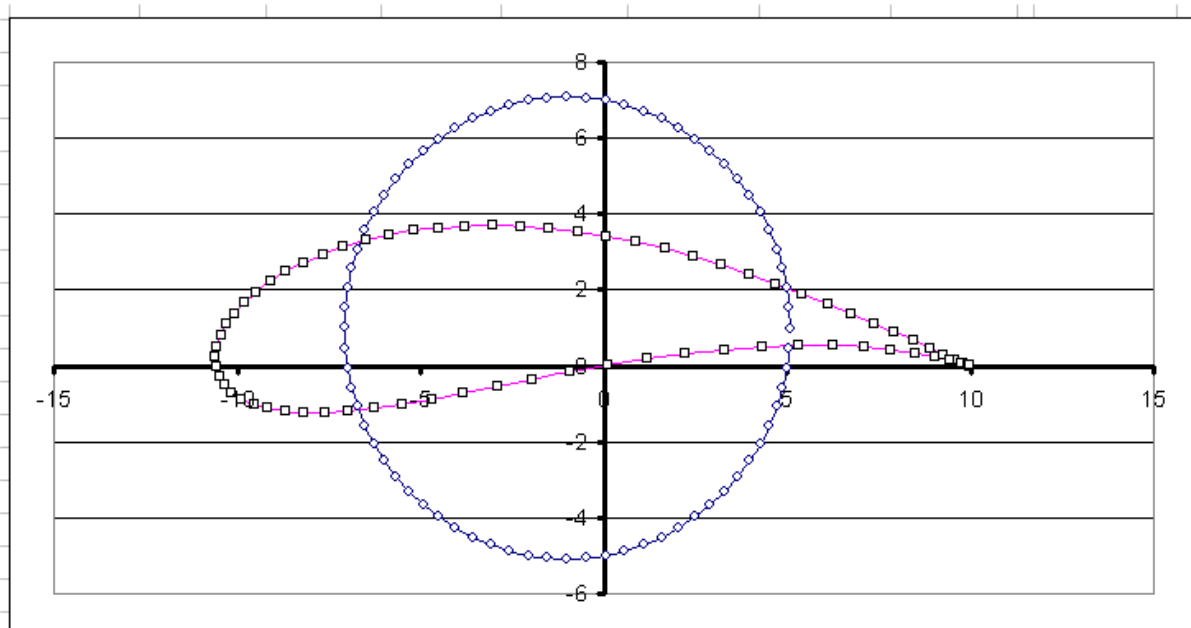


Figure 1: Base form is the circle  $M(-1;+1)$  and  $r = 37^{0.5}$   
The profile results from the conformal mapping according to the formulae  $u = \dots$  and  $v = \dots$



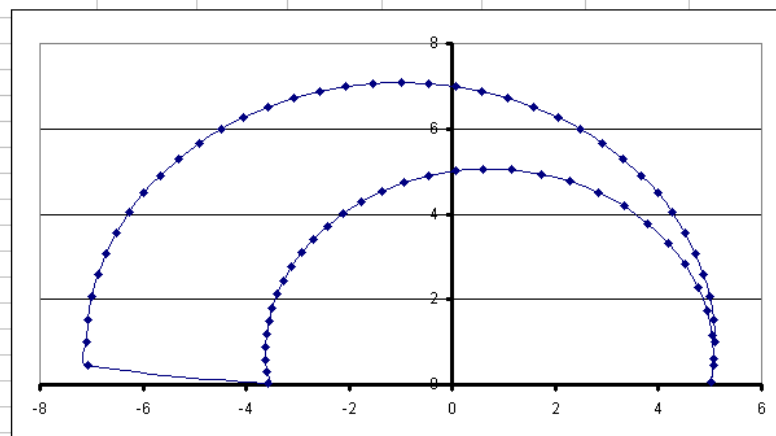


Figure 2 : This the back transformation applying the 4th DERIVE solution  
The upper circle is the right one!

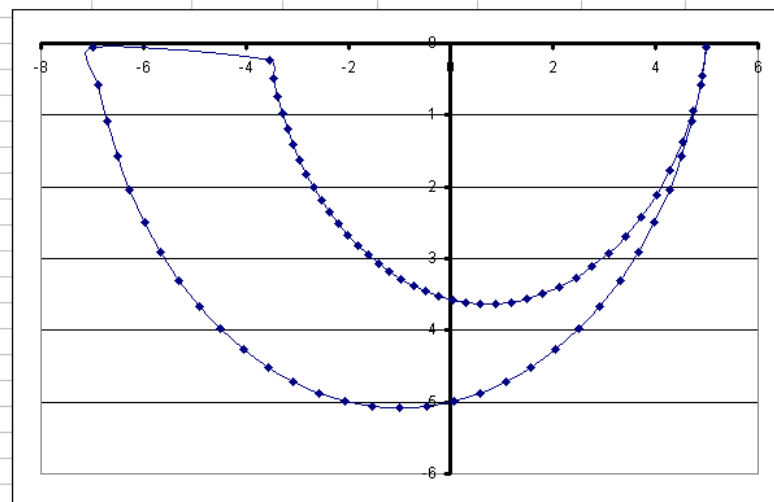
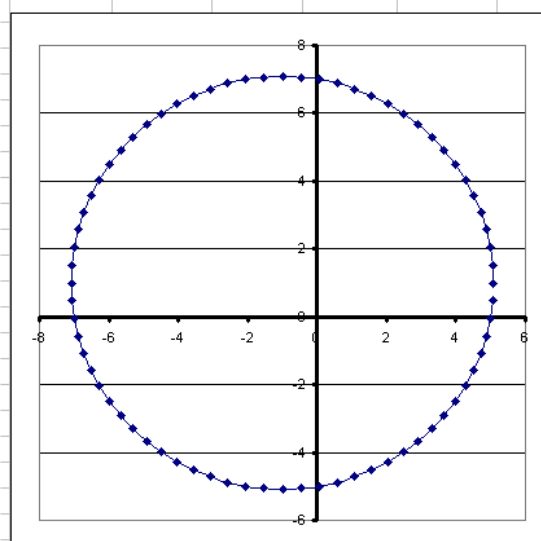


Figure 3 : Back transformation using DERIVE's 3rd solution  
Now the lower circle is the correct one!

For  $0^\circ$  to  $185^\circ$  the results of the 4th solution are fitting exactly to the given circle.  
The graph shows the combination of both series of data.



One could be happy, but why do not fit the results for  $355^\circ$  and  $360^\circ$ ?

**Values for  $355^\circ$  would be fine if we would take the 4th solution!**

I define four functions for the nontrivial solutions (sol1 and sol2 are the complex ones, sol3 and sol4 the real ones).

sol3(u, v, k) :=

$$\left[ \frac{\sqrt{2} \cdot ((\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^4 + 2 \cdot u^2 \cdot v + v^4)} - 4 \cdot k + u^2 - v^2) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^4 + 2 \cdot u^2 \cdot v + v^4)} - 8 \cdot u \cdot v)}{8 \cdot u \cdot v} \right. \\ \left. \cdot \frac{(u^2 \cdot v^2 + v^4) + 4 \cdot k - u^2 + v^2 + 2 \cdot \sqrt{2 \cdot u^2 \cdot v}}{4} + \frac{v}{2} \right]$$

sol4(u, v, k) :=

$$\left[ \frac{\sqrt{2} \cdot (2 \cdot \sqrt{2 \cdot u^2 \cdot v} - (\sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^4 + 2 \cdot u^2 \cdot v + v^4)} - 4 \cdot k + u^2 - v^2) \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^4 + 2 \cdot u^2 \cdot v + v^4)} - 8 \cdot u \cdot v)}{8 \cdot u \cdot v} \right. \\ \left. \cdot \frac{(u^2 + u^4 + 2 \cdot u^2 \cdot v + v^4) + 4 \cdot k - u^2 + v^2}{4}, \frac{v}{2} - \frac{\sqrt{2} \cdot \sqrt{(16 \cdot k^2 + 8 \cdot k \cdot (v^2 - u^2) + u^4 + 2 \cdot u^2 \cdot v + v^4)} + 4 \cdot k - u^2 + v^2}{4} \right]$$

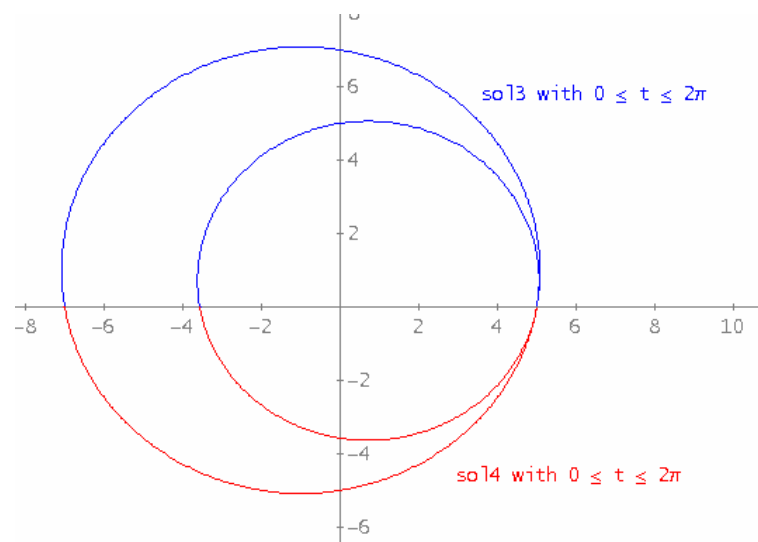
I try the functions with  $u$  and  $v$  being the parameter form of the wing – and  $k = 25$  – and hope to come back to the circle:

$$u_- := \frac{2 \cdot (37 \cdot \cos(t))^2 - \sqrt{37 \cdot \cos(t)} \cdot (\sqrt{37 \cdot \sin(t)} + 33) + \sqrt{37 \cdot \sin(t)} + 32}{2 \cdot \sqrt{37 \cdot \cos(t)} - 2 \cdot \sqrt{37 \cdot \sin(t)} - 39}$$

$$v_- := \frac{2 \cdot (\sqrt{37 \cdot \sin(t)} + 1) \cdot (\sqrt{37 \cdot \cos(t)} - \sqrt{37 \cdot \sin(t)} - 7)}{2 \cdot \sqrt{37 \cdot \cos(t)} - 2 \cdot \sqrt{37 \cdot \sin(t)} - 39}$$

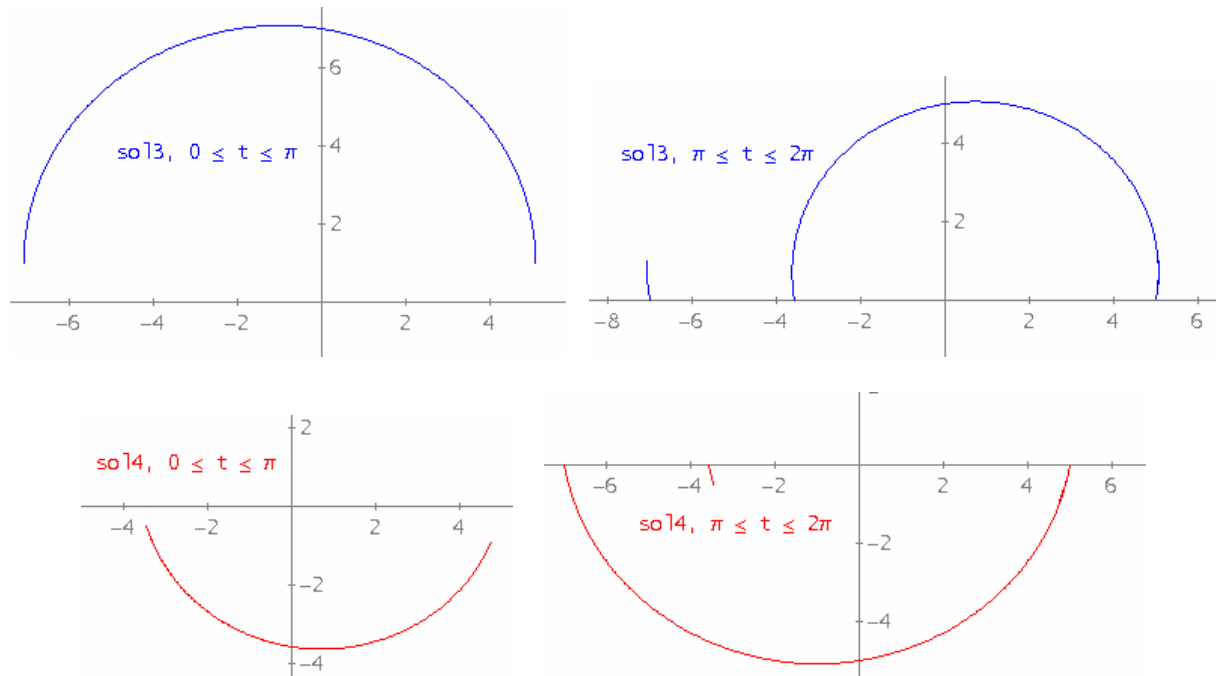
sol3(u<sub>-</sub>, v<sub>-</sub>, 25)

sol4(u<sub>-</sub>, v<sub>-</sub>, 25)



This is interesting and surprising as well. I don't receive the expected circle but I receive two circles, one of them being the expected one. At the first view it seems to be that I am not facing Klaus Körner's sign-problem.

But see what is happening when plotting the circles in steps:



There are strange arcs belonging to the other circle which I don't know how to explain. (I must admit that I am not an expert in conformal mappings.)

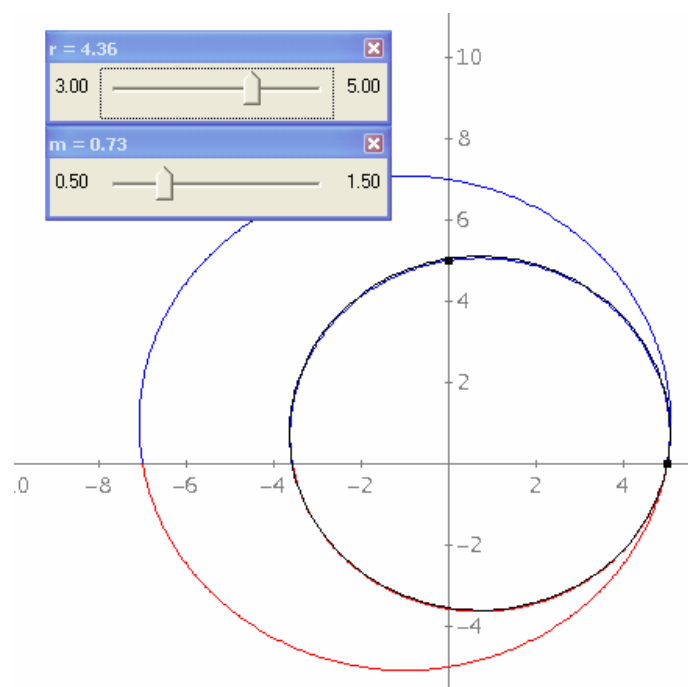
Now I'll deal with the second circle. Its centre seems to lay on  $y = x$  and it passes (5,0) and (0,5). This is not enough to find its parameters. I didn't know better than to use sliders for estimating the coordinates of the centre and the radius.

Centre: ( $\approx 0.73$ ,  $\approx 0.73$ )

Radius  $\approx 4.36$

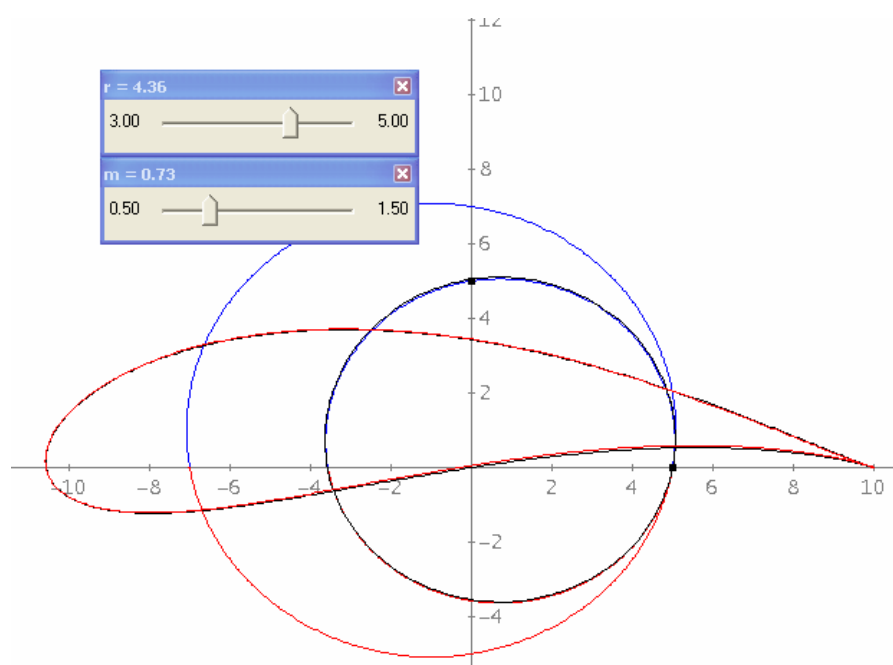
What is the result when applying the conformal mapping from above to this (second) circle?

Let's see:



$$\text{SUBST}\left(\left[\left[x \cdot \left(1 + \frac{25}{x^2 + y^2}\right), y \cdot \left(1 - \frac{25}{x^2 + y^2}\right)\right], [x, y], [4.36 \cdot \cos(t) + 0.73, 4.36 \cdot \sin(t) + 0.73]\right)\right]$$

$$\left[ \frac{0.01 \cdot (436 \cdot \cos(t) + 73) \cdot (3.1828 \cdot 10^4 \cdot \cos(t) + 3.1828 \cdot 10^4 \cdot \sin(t) + 2.25377 \cdot 10^5)}{3.1828 \cdot 10^4 \cdot \cos(t) + 3.1828 \cdot 10^4 \cdot \sin(t) + 1.00377 \cdot 10^5}, \right. \\ \left. \frac{0.01 \cdot (436 \cdot \sin(t) + 73) \cdot (3.1828 \cdot 10^4 \cdot \cos(t) + 3.1828 \cdot 10^4 \cdot \sin(t) - 2.4623 \cdot 10^4)}{3.1828 \cdot 10^4 \cdot \cos(t) + 3.1828 \cdot 10^4 \cdot \sin(t) + 1.00377 \cdot 10^5} \right]$$



Obviously we receive the same profile as result of the mapping applied to two different circles (which are surely related in any form, but which one? Who knows?)

(It is also possible to generate trochoids in two different ways.)

I am much more surprised by the two little pieces of the circles. Do you have any explanation for this? Who is an expert?

Josef

Just now, finalizing this DNL, I performed the calculations with *MATHEMATICA* and received the same results – some in another form. I found that the *DERIVE* results are much more compact. You can find parts of the *MATHEMATICA* treatment after Mr Körner's next mail.

And I had one more question which I tried to answer for myself – and for you, too, of course.

But see first Klaus Körner's response:

Hello Mr Böhm,

I would like to inform you about an interesting link:

<http://www.excelformeln.de/uberuns.html>

You can find numerous meaningful (and also less meaningful) EXCEL-applications (free of charge). Mr Klaus Kühnlein is the main organizer of this publication.

Finally I'd like to thank for your efforts. Your results agree completely with my Excel-solutions. I will try to "back transform" the profile of a real turbine blade in order to find out whether it results as a more or less regular figure.

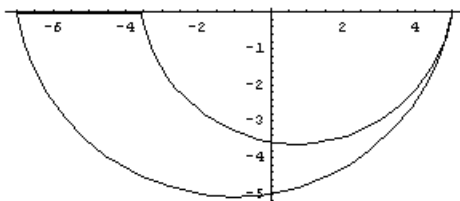
...

Best regards

Klaus Körner

These are the *MATHEMATICA*-calculations and plots.

```
In[10]:= ParametricPlot[sol3[a, b, 25], {t, 0, 2*Pi}, AspectRatio -> Automatic]
```

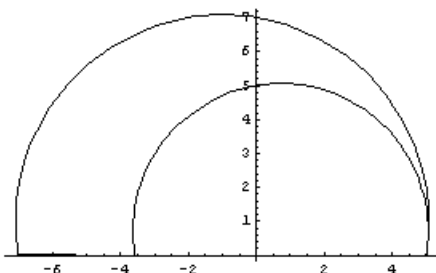


Out[10]= -Graphics -

```
In[11]:= sol4[u_, v_, k_] :=
```

$$\left\{ \frac{1}{8 u v^2} \left( -2 \sqrt{2} \sqrt{-u^2 + v^2 + 4 k} + \sqrt{16 k v^2 + (u^2 + v^2 - 4 k)^2} v^3 + 4 u^2 v^2 + \sqrt{2} \left( -u \right. \right. \right. \\ \left. \left. \left. 2 \sqrt{2} u^2 \sqrt{-u^2 + v^2 + 4 k} + \sqrt{16 k v^2 + (u^2 + v^2 - 4 k)^2} v - 8 \sqrt{2} k \sqrt{-u^2 + v^2 + 4 k} \right) \right. \right. \\ \left. \left. \frac{1}{16} \left( 8 v + 4 \sqrt{2} \sqrt{-u^2 + v^2 + 4 k} + \sqrt{16 k v^2 + (u^2 + v^2 - 4 k)^2} \right) \right) \right\}$$

```
In[12]:= ParametricPlot[sol4[a, b, 25], {t, 0, 2*Pi}, AspectRatio -> Automatic]
```



At first I performed the *MATHEMATICA*-plots using  $u_-$  and  $v_-$  from page 32. Then I asked *MATHEMATICA* to do the substitution and received quite other expressions ( $c$  and  $d$  in In[16]).

In[15]=  $\{x \star (1 + 25 / (x^2 + y^2)), y \star (1 - 25 / (x^2 + y^2))\} /. \{x \rightarrow \text{Sqrt}[37] \star \text{Cos}[t] - 1, y \rightarrow \text{Sqrt}$

Out[15]=  $\left\{(\sqrt{37} \cos(t) - 1) \left(1 + \frac{25}{(\sqrt{37} \cos(t) - 1)^2 + (\sqrt{37} \sin(t) + 1)^2}\right), (\sqrt{37} \sin(t) + 1) \left(1 - \frac{25}{(\sqrt{37} \cos(t) - 1)^2 + (\sqrt{37} \sin(t) + 1)^2}\right)\right\}$

In[16]=  $c = (\sqrt{37} \cos(t) - 1) \left(1 + \frac{25}{(\sqrt{37} \cos(t) - 1)^2 + (\sqrt{37} \sin(t) + 1)^2}\right);$

$d = (\sqrt{37} \sin(t) + 1) \left(1 - \frac{25}{(\sqrt{37} \cos(t) - 1)^2 + (\sqrt{37} \sin(t) + 1)^2}\right);$

But the resulting plot is pretty the same.

It might be a good exercise for students proofing the identity of  $c$  and  $u_-$  ( $d$  and  $v_-$ ). For us it is easier, let's ask DERIVE!

$$c := (\sqrt{37} \cdot \cos(t) - 1) \cdot \left(1 + \frac{25}{(\sqrt{37} \cdot \cos(t) - 1)^2 + (\sqrt{37} \cdot \sin(t) + 1)^2}\right)$$

Precision := Exact

Notation := Rational

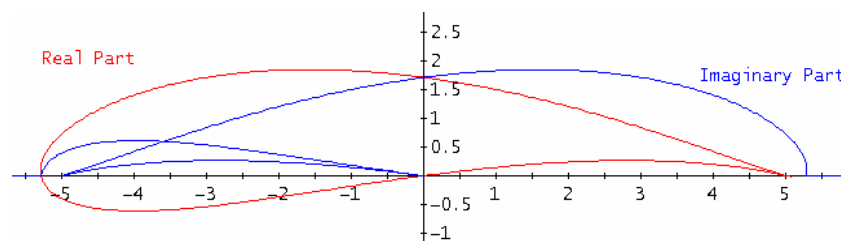
$u_- - c = 0$

Solutions #1 and #2 are complex, back transformation applying them gives also complex expressions and we cannot obtain any plot.

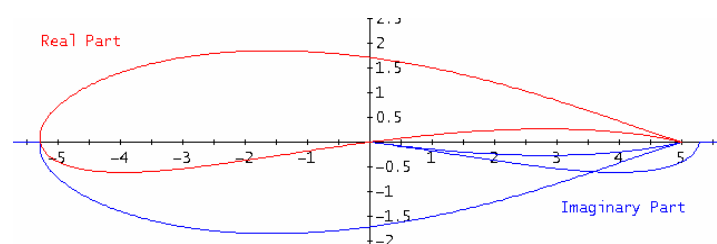
I wondered what the plot of the real part (and the imaginary part, too, of course) would look like and I was presented a big surprise:

RE(sol1( $u_-$ ,  $v_-$ , 25))

IM(sol1( $u_-$ ,  $v_-$ , 25))



The real part was again the basic wing profile! Solution #2 gave also the wing, the imaginary part is mirrored wrt to the origin:



## From the eDUG-Forum: Tensor Algebra and DERIVE DOS

**Thomas Fowler:**

I posted two new versions of the Tensor Algebra file, updating the fine work done by Hans Dudler. The version for Derive 5 & 6 is TensorAlgebraExtd.mth. The version for Derive 3 (last DOS release) is TENSORNE.MTH. I also posted an updated and expanded explanation of the operations, TensorAlgebra.doc. The extensions are primarily for functions in use in the study of general relativity, and include the Riemann Curvature function K.

Tom

(See pages 39 – 41)

**Rick Nungester**

I have Derive for DOS v3.04 working on Windows XP, but it always starts full-screen instead of in a small terminal-like text window. Right-clicking on the shortcut and seeing its Properties shows tabs General, Program, Font, Memory, Screen, Misc, Compatibility, and Summary. I've played with various settings, but it always starts full-screen.

Can Derive for DOS be made to open in a small window instead of full-screen?

I also have Derive for Windows v5.06, but use Derive for DOS v3.04 on my circa-1994 HP 200LX palmtop and want the same functionality on Windows.

Rick

**Thomas Fowler**

Until TI decides to issue a version of DERIVE for Android devices (which would be a cash cow for them), you can still get most of the functionality if you have your old DOS release. The last DOS release was DERIVE 3. I have it running on an Android tablet under aDOSBox. It works well, but you really need an external keyboard. You can speed it up by going into the menu option (from the F1 key), selecting DOSBox, and increasing the CPU cycle variable. I am running mine at 87350, about 20x faster than the default, which naturally emulates and 8088.

Tom

**Aleksey Tetyorko**

IMHO The last DOS release was DERIVE 4.11 with the extended syntax.

Sorry, one question: can you switch to a windowed mode by pressing Alt-Enter? I cannot! My DOS Derives are 3.11 and 4.11. But when I've changed the Derive Display Mode to Text (Option, Display, Mode:Text), I made my Derive into a window by Alt-Enter. Do your Derive and XP behave in this manner?

Aleksey

**Rick Nungester**

Thank you Aleksey, that is just what I needed. Now I can start Derive for DOS 3.04 from a Desktop icon directly into a window without using Alt-Enter at all. Here is what I did:

In Derive, make Text mode the start-up setting:

- \* Start Derive (no other changes before the next steps)
- \* Option, Display, Mode:Text
- \* Transfer, Save, State:DERIVE.INI (Y, overwrite)
- \* Quit

In Windows XP, make "window" the start-up choice:

- \* Desktop icon, right-click, Properties, Screen tab, Usage:Window, OK

Now whenever I double-click the Desktop icon, Derive for DOS 3.04 starts in a text window. Plotting still works, drawing lines as a series of "." text characters instead of using graphics. But then switching to Option, Display, Mode:Graphics switches to full-screen and good-looking graphics. If desired, then switching back to text mode and doing Alt-Enter will go from nice full-screen graphics back to a smaller text-only window. Great!

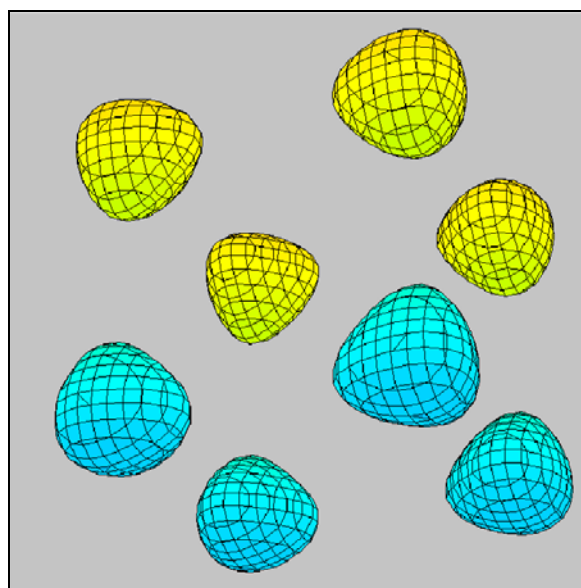
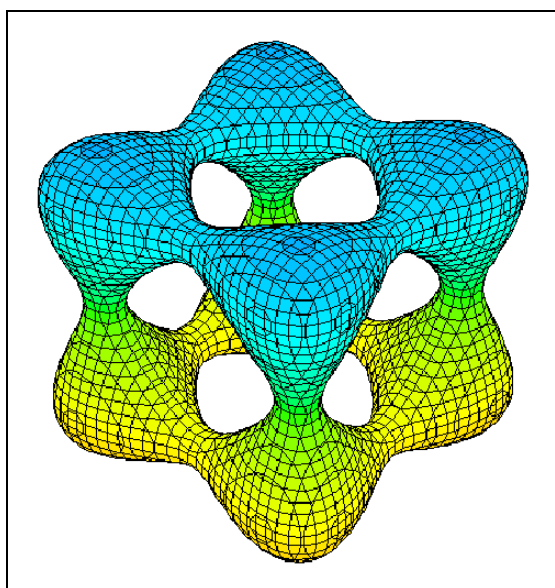
Rick

**Aleksey Tetyorko**

For your service. You may try using DosBox (DOS emulator for XP), and you can do it without switching to Derive text mode. I've test this method some hours before now. My DosBox version is 0.73, maybe not the last. I cannot recommend it because of slow operation.

Aleksey

When informing about tensors in E. Weisstein's *Concise Encyclopedia of Mathematics* I came across this pretty quartic surface, a TANGLECUBE, Josef



$$x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 11.8(15) = 0$$



\*\*\*\*\*  
**\* The Derive Tensor Algebra and Analysis Package \***  
**\* Documentation for the utility file Tensor.mth \***  
 \*\*\*\*\*

Written by Hans A. Dudler (619 420-1787) 25 November 1994

Updated for Derive 5

10 November 2001

New functions added, documentation expanded by Tom Fowler, September 2011

This file describes the DERIVE utility file Tensor.mth which provides definitions and functions for tensor algebra and analysis. Tensor.mth should normally be loaded as a utility file using the File > Load > Utility File command. Note that Tensor.mth activates the case sensitive and word input modes. As an introduction to the concepts described in this document the demonstration files Tensor1.dmo and Tensor2.dmo are recommended. They should be loaded using the File > Load > Demo File command.

## 1. Tensor Representation in DERIVE.

=====

From the algebraic point-of-view tensors are an extension of the vector/matrix concept to higher order arrays. Thus a tensor of rank (order) 1 is a vector. A rank 2 tensor is a matrix, which DERIVE treats as a vector of vectors. A rank 3 tensor is a vector of matrices, a rank 4 tensor is a matrix of matrices, and so on. The obvious way to handle tensors in DERIVE is therefore to extend the vector concept to higher ranks (orders).

The equivalent of the ELEMENT function to extract an element from a tensor is the function (see function descriptions below)

EL\_T(A,iv) ,

where A is the name of the tensor and iv is the index 'vector'(quotes used to distinguish a DERIVE vector from a rank 1 tensor). As an example consider the tensor of rank (order) 3 and dimension 3

$$A = \left[ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} \begin{bmatrix} s & t & u \\ v & w & x \\ y & z & 0 \end{bmatrix} \right]$$

Then

EL\_T(A,[2,3,2]) returns q

EL\_T(A,[3,1,3]) returns u

As is clear from the example, the first index (element 1 of the index vector) selects the matrix, the second index selects the row-vector in the selected matrix and the third index finally selects the element in the row vector. Note that under DERIVE the indices must range from 1 to n (the space dimension), zero (often used in relativity) is not allowed. From the above example we deduce that

EL\_T(A,[i,j,k]) <=> ELEMENT(ELEMENT(ELEMENT(A,i),j),k)

in this case, and the extension to higher ranks is obvious. Note that

EL\_T(A,[i]) <=> ELEMENT(A,i)

CURVE\_T(T,Γ,s) : vector components needed for curvature and geodesics, in terms of arc length parameter s. Calculates components of **b** as

$$\mathbf{b} = \left( \frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds} \right), \quad \mathbf{x} = (x^i(t)), \quad \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds}$$

And arc length parameter  $s=s(t)$  given by

$$s(t) = \int_a^t \sqrt{\varepsilon g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}} du$$

Easiest to understand with an example. First define G (metric tensor), T (curve whose curvature is desired), and coordinate vector x:

$$\#101: \left[ \mathbf{x} := [\mathbf{x1}, \mathbf{x2}], \mathbf{G} := \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x1}^2 \end{bmatrix}, \mathbf{T} := [\mathbf{a} \cdot \text{SEC}(\mathbf{t}), \mathbf{t}] \right]$$

Next compute Christoffel Symbols of 2<sup>nd</sup> Kind:

$$\#102: \mathbf{G\_} := \text{CHRIS2}(\mathbf{G}, \mathbf{G}^{-1}, \mathbf{x})$$

$$\#103: \mathbf{G\_} := \left[ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\mathbf{x1}} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{\mathbf{x1}} \\ -\mathbf{x1} & 0 \end{bmatrix} \right]$$

Now convert G to parameterized coordinates by Simplify, Variable Substitution, and calculate arc length parameter s. Replace u with t, then solve for t in terms of s:

$$\#103: \mathbf{G\_} := \left[ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\mathbf{x1}} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{\mathbf{x1}} \\ -\mathbf{x1} & 0 \end{bmatrix} \right]$$

$$\#104: \mathbf{G} := \begin{bmatrix} 1 & 0 \\ 0 & \frac{\mathbf{a}^2}{\cos(\mathbf{t})^2} \end{bmatrix}$$

$$\#105: \text{ARC\_LENGTH\_P}(\mathbf{T}, \mathbf{G}, \mathbf{t}) = |\mathbf{a}| \cdot \text{TAN}(\mathbf{u})$$

$$\#106: \text{SOLVE}(s = |\mathbf{a}| \cdot \text{TAN}(\mathbf{t}), \mathbf{t}, \text{Real})$$

$$\#107: \mathbf{t} = \text{ATAN}\left(\frac{s}{|\mathbf{a}|}\right) - \pi \vee \mathbf{t} = \text{ATAN}\left(\frac{s}{|\mathbf{a}|}\right) + \pi \vee \mathbf{t} = \text{ATAN}\left(\frac{s}{|\mathbf{a}|}\right)$$

$$\#108: \mathbf{T} := \left[ \mathbf{a} \cdot \text{SEC}\left(\text{ATAN}\left(\frac{s}{\mathbf{a}}\right)\right), \text{ATAN}\left(\frac{s}{\mathbf{a}}\right) \right]$$

$$\#109: \mathbf{T} := \left[ \sqrt{(\mathbf{a}^2 + s^2)} \cdot \text{SIGN}(\mathbf{a}), \text{ATAN}\left(\frac{s}{\mathbf{a}}\right) \right]$$

In this case we have chosen the last of the three possibilities for  $t$ , and assumed that  $a > 0$ , giving  $t = \text{ATAN}(s/a)$ , and converting  $T$  to  $s$  variable by substituting this and invoking Simplify, Basic. This allows us to get rid of the SIGN function, so we have

$$\#110: T := \left[ \sqrt{a^2 + s^2}, \text{ATAN}\left(\frac{s}{a}\right) \right]$$

Use these values for  $x_i$ 's in terms of  $s$  to substitute into  $G_-$ :

$$\#111: G_- := \left[ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{a^2 + s^2}} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{\sqrt{a^2 + s^2}} \\ -\sqrt{a^2 + s^2} & 0 \end{bmatrix} \right]$$

Invoking CURVATURE on this will yield 0, which indicates that the original curve was a geodesic.

$$\#112: \text{CURVE\_T}(T, G_-, s) = [0, 0]$$

(There are problems with the "T" in the DERIVE-file because  $\Gamma$  is reserved for the Gamma-function. I took  $G_-$  instead, Josef)

#### About Tensors (from *CRC Concise Encyclopedia of Mathematics*):

An  $n$ th-rank tensor is a mathematical object in  $m$ -dimensional space which has  $n$  indices and  $m^n$  components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. If the components of any tensor of any rank vanish in one particular coordinate system, they vanish in all coordinate systems.

Tensors of rank 0 are called scalars, tensors of rank one are vectors and tensors of rank two are matrices, ...

Two recommended links to introductory papers on Tensors

An Introduction to Tensors for Students of Physics and Engineering (NASA):

[http://www.grc.nasa.gov/WWW/K-12/Numbers/Math/documents/Tensors\\_TM2002211716.pdf](http://www.grc.nasa.gov/WWW/K-12/Numbers/Math/documents/Tensors_TM2002211716.pdf)

A paper in GERMAN (Technische Universität Graz):

[http://www.math.tugraz.at/~ganster/lv\\_vektoranalysis\\_ss\\_10/17\\_tensor\\_definition.pdf](http://www.math.tugraz.at/~ganster/lv_vektoranalysis_ss_10/17_tensor_definition.pdf)

(GERMAN)

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Von: **R.N. Folsom** [rnfolson@redshift.com]  
 Gesendet: Donnerstag, 16. Dezember 2010 09:24  
 An: "Böhm, Josef"  
 Betreff: **A Constrained Inequality's Truth**

Josef:

First, thank you very much for your message of 13 December. I very much appreciate your work supporting the Derive User Group, and me.

I have attached a zipped file, "GrowthRates Summed vs Multiplied, N3 - Derive," that contains a dfw file of the same name, which describes a problem I am having. I hope you can pass the file on to someone who might know the answer, or that you decide it is worth including in the next newsletter bulletin.

But I would be happy to revise this document if you think I should do so.

Sending the Comparative Statics Technique information is definitely on my short term todo list.

Thanks again for your help.

As you will see below, I am trying to formally prove the circumstances under which a particular inequality is true --- and Derive's Simplify or Solve command actually says that it is true. (For now, all I need is sufficient conditions.)

But I have not been able to figure out the Derive6.1 command or commands that will do that.

My sources are Bernhard Kutzler and Vlasta Kolol-Voljc's "Introduction to Derive 6, Advanced Mathematics for Your PC" (hereafter Derive6); and Albert and Joan Rich, Theresa Shelby, and David Stoutemeyer's "Derive User Manual Version 6" (hereafter Derive3).

Derive6's equation solver (page 57), when used to determine whether a relationship is True or False, doesn't apply to what I am trying to do because it doesn't allow conditions to be imposed on the equation's (or, in my case, the inequality's) variables.

Derive6's discussion of the IF tool (on pages 166 and 227) didn't help because I didn't understand page 166 (I don't know any Analytic Geometry), and although I think that I did understand page 227 I couldn't figure out how to apply it to my problem.

Derive3's discussion of the IF tool (pages 309-310), made sense, but again I couldn't figure out how to apply it to my problem.

My understanding is that the IF tool has four fields: a test condition, a "then" clause of results if the test condition is satisfied, an "else" clause of results if the test condition is NOT satisfied, and an "unknown" clause.

Derive3's opening IF example (page 309) is a payroll problem:  $IF(h \leq 40, 10h, 400 + 15(h - 40))$  where  $h$  is hours worked and  $10h$  and  $15(h - 40)$  are worker payment amounts. Unfortunately, the problem does not include the unknown clause, which if not included but is needed turns out to be the entire IF(...) tool. An example that does include an unknown clause would have been helpful.

What I want is a tool, or set of tools, that let me state restrictions on my inequality's variables, then state the inequality itself, and returns either "true" or "false", or ? (or the same object) if the restrictions are too weak to determine whether the inequality is true or false.

The  $r_{\downarrow n}$  variables below are annual (or some other time period) rates of growth, such as interest rates, inflation and deflation rates, real output growth or decline rates, etc. Rates of growth  $r_{\downarrow n}$  can be either positive, zero, or negative.

The number N is the number of time periods being considered. Sometimes "total" rates of growth over N time periods are calculated by summing the N periodic rates of change  $r_{\downarrow n}$  ( $n = 1, 2, \dots, N$ ), which is less accurate than multiplying the N corresponding  $(1 + r_{\downarrow n})$  factors. Summing the  $r_{\downarrow n}$  can give results that are either larger or smaller than the more accurate multiplication of the  $(1 + r_{\downarrow n})$ .

However, a common situation is that over N time periods, more rates of growth  $r_{\downarrow n}$  are positive than negative. If that difference is sufficiently large, summing the N periodic rates of change  $r_{\downarrow n}$  gives a lower total rate of growth over N periods than does multiplying the corresponding  $(1 + r_{\downarrow n})$  numbers, so that summation understates the true total rate of change over N periods.

Mathematically, that "common situation" inequality can be written in three basic ways. See statements #2, #3, and #4, below.

SATISFYING THE "COMMON SITUATION" INEQUALITY; A SUFFICIENT CONDITION: All  $r_{\downarrow n} \geq 0$

If ALL of the  $r_{\downarrow n}$  are positive or zero ( $r_{\downarrow n} \geq 0$ ), the "common situation" inequality holds, because the left side's terms are smaller than the right side's factors. For an example, let  $N = 3$ , and see statements #5, #6, #7, and #8. Note that in #7 --- a rearrangement of #2 with  $N = 3$  --- the left side is -1, and ALL of the variables on the right side are positive.

Therefore, a SUFFICIENT condition for the "usual condition" inequality is simply that all  $r_{\downarrow n} \geq 0$ .

#1: CaseMode := Sensitive

$$\#2: \sum_{n=1}^N r_{\downarrow n} < \prod_{n=1}^N (1 + r_{\downarrow n})$$

$$\#3: 0 < \left( \prod_{n=1}^N (1 + r_{\downarrow n}) \right) - \sum_{n=1}^N r_{\downarrow n}$$

$$\#4: -1 < \left( \prod_{n=1}^N (1 + r_{\downarrow n}) \right) - 1 - \sum_{n=1}^N r_{\downarrow n}$$

For example, if  $N = 3$ :

$$\#5: -1 < \left( \prod_{n=1}^3 (1 + r_{\downarrow n}) \right) - 1 - \sum_{n=1}^3 r_{\downarrow n}$$

Statement #5 can be written as

$$\#6: -1 < (r_{\downarrow 1} + 1) \cdot (r_{\downarrow 2} + 1) \cdot (r_{\downarrow 3} + 1) - 1 - (r_{\downarrow 3} + r_{\downarrow 2} + r_{\downarrow 1})$$

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In statement #6, multiplying out the  $(r_{\downarrow n} + 1)$  factors would eliminate the  $-1 - (r_{\downarrow 3} + r_{\downarrow 2} + r_{\downarrow 1})$  expression. Doing that (by using Derive's Simplify-Factor command) gives statement #7.

$$\#7: \quad -1 < r_{\downarrow 3} \cdot (r_{\downarrow 2} \cdot (r_{\downarrow 1} + 1) + r_{\downarrow 1}) + r_{\downarrow 1} \cdot r_{\downarrow 2}$$

or

$$\#8: \quad -1 < r_{\downarrow 1} \cdot r_{\downarrow 2} \cdot r_{\downarrow 3} + r_{\downarrow 2} \cdot r_{\downarrow 3} + r_{\downarrow 1} \cdot r_{\downarrow 3} + r_{\downarrow 1} \cdot r_{\downarrow 2}$$

Note: I have done the same calculations above using  $N = 5$ .

Given the sufficient condition that all  $r_{\downarrow n} \geq 0$ , statements #7 and #8 are obviously true because the left side is negative, and every term on the right side is positive.

And since statements #7 and #8 are merely a rearrangement of #5, which is an example of statements #4, #3, and #2, statement #7 demonstrates that if all  $r_{\downarrow n} \geq 0$ , the "usual situation" inequality holds, so that summing  $N$  periodic rates of change  $r_{\downarrow n}$  (instead of multiplying corresponding  $1 + r_{\downarrow n}$  numbers) understates the true total rate of change over  $N$  periods.

HOWEVER, I need to learn how to persuade Derive6.1 to do a formal proof (results either true, false, or unknown) of the preceding paragraph's results, because my next step is to experiment with less restrictive conditions, such as all  $r_{\downarrow n} \geq -1$ .

=====

TO DO: Figure out a formal proof --- for any statement #5 through #8 --- that when #10 (below) holds, comes up with "true" rather than the wrong answer of "false."

For example, Simplify-Basic of either IF(#10, #7) or IF(#7, #10) returns the same (written out) IF statement, presumably meaning that Simplify could not determine whether the inequality was true or false, despite the "all  $r_{\downarrow n} \geq 0$ " variable conditions.

But Solve-Expression of either IF(#10, #7) or IF(#7, #10) concludes that the constrained inequality is false, when I know that it is true (from the analysis that follows #8).

I clearly am not using IF properly. It may not be intended for what I am trying to do. In that case, I need to know what else to do.

$$\#9: \quad \text{VECTOR}(r_{\downarrow n} \geq 0, n, 1, 3, 1)$$

$$\#10: \quad \left[ r_{\downarrow 3} \geq 0 \wedge r_{\downarrow 2} \geq 0 \wedge r_{\downarrow 1} \geq 0 \right]$$

Roger Folsom

Thanks for welcoming me to the Derive Users Group.

Actually, it turns out (prior to the Derive 6.1 CD) that although my latest MS-DOS diskettes were for Derive 4.0.7, I have a Derive 3 manual (in addition to the Derive 6 manual by Bernhard Kutzler and Vlasta Kokol-Voljc), so apparently my use of Derive goes back at least to Derive 3.x. The Derive 3 manual is quite useful in providing more detail about various Derive features and capabilities.

> Applications in Economics would be highly appreciated!! <

You may already have some. I have, on paper, a message posted by "R.N. Folsom" on the Derive BBS, 26 April 1993, about "Folsom's Comparative Statics Technique." I was responding to a message #2896, 19 April 1993, on "Derive/Mathware" (apparently a different BBS) by "Harald Lang to Roger Folsom," with the title "Harald Lang's Use of Taylor's Series to Specify the Chain Rule." And that was preceded, again on "Derive/Mathware," by message #2893 by "Hadud to Roger Folsom about Response To Questions," which apparently was a response to a set of 7 questions posted by me. Unfortunately, I don't have the seven questions, but Hadud's answers give pretty good clues about what my questions were. However, they apparently were about how Derive worked, and did not have anything to do specifically with economics.

But "Comparative Statics" is something that economists do frequently. If you have a system of functions --- i.e., a model --- describing a set of markets (as in microeconomics) or describing in broad terms the entire economy (macroeconomics), the variables fall into two categories: endogenous (whose values are determined by the model's functions) and exogenous (whose values are determined outside of the model, e.g. Russia's awful wheat-destroying fires this past summer). And you want the model to tell you what the exogenous event will do to markets (not merely the market for wheat, but also to markets for grain substitutes such as corn, other things that grow such as cotton, markets for beef since cattle often are fed corn, etc.). In short, you want to know (more accurately, to estimate) the effects of an exogenous variable (or possibly multiple exogenous variables) on endogenous variables such as the prices and quantities of corn, cotton, beef, etc.

Comparative statics usually (maybe always) involves the "chain rule": derivatives (or finite difference quotients) of  $y = Y(x)$  and  $x = X(a)$  where  $x$  and  $y$  are endogenous and  $a$  is exogenous, and where I have used capital letters instead of "f" to indicate a function (because I'll probably want to use  $f$  for something else, such as fire severity!). In short, you want to know not merely  $dy/dx$  and  $dx/da$  but also  $dy/da = [dY(x)/dx].[dX(a)/da]$ , which in the U.S. and maybe elsewhere is commonly known as the "chain rule."

But for a multi-function system, to calculate a full set of total derivatives of all of the endogenous variables with respect to each of the exogenous variables, Derive wouldn't use the chain rule. Harald Lang came up with one solution, and I used his notation to come up with another solution (namely, not substituting any function into any other function before taking derivatives and calculating Jacobian matrices for the system).

If you do not already have the messages I have described above, let me know and I will send them. I'd do so right now except that they all are paper documents and the ink is not as dark as I wish it were, so I would need some time to fiddle around with my scanner and optical character recognition software to send them electronically. I don't have time to do that right now because I am facing deadlines for two entirely different projects, but I think I could get to it in January.

My apologies for taking your time in trying to "explain" the terms comparative statics, and endogenous and exogenous. I did so because I don't know if those terms are used by non-economist mathematicians, or in countries outside the U.S. and Canada.

I briefly visited your Didactics of Computer Algebra <http://www.acdca.ac.at/>, and tentatively concluded that it focuses on teaching computer algebra. But the site's index frequently uses the CAS acronym, where presumably C is for Computer and A is for Algebra, but I'm wondering what the S is for. Studies? Or a non-English word? At your convenience, please satisfy my curiosity!

A couple of suggestions: After downloading all of the zipped MTH files from the DUG website, one at a time, I then saw and followed the recommendation to look at the "back" of the PDF membership file, which contained the statement "Disk of the year containing all .MTH files is included." It's been a long day (almost 2:00 a.m. California time), so I likely am operating in stupid mode, but: In what is that "Disk" included?

With or without a disk, it might be useful if the DUG website had three large zip files that included each of the three groups of numbered MTH zip files.

Similarly, since the newsletters/bulletins apparently include articles that explain the use of those zipped mth files, it might be useful if the DUG website had zipped packages of dnl\*.pdf files, so each pdf file wouldn't have to be downloaded individually.

I'll close with a very minor request: Mathematicians, and also Derive, like to write polynomial functions "downhill," meaning with the largest exponent term first, followed by successively smaller exponent terms, and then a constant. I do understand that that sequence has mathematical advantages, for example by facilitating dividing one polynomial by another. But economists often like to write polynomials "uphill" (constant term first, followed by terms with successively larger exponents), because we often don't know how big the polynomial needs to be until we use data to estimate its coefficients. And it is more convenient to add or delete a high-order term if it is at the end of the polynomial rather than at its beginning, because if you add or delete a term at the beginning of the polynomial the coefficients of the previous version (if identified by subscripts) have to be renumbered.

I wouldn't want to change the way Derive works. But if you know of a Derive operation (probably created by some user, since I'm pretty sure it's not in Derive itself) that rearranges a downhill polynomial (and any polynomial subcomponents) and rewrites it (and its subcomponents) to go uphill rather than downhill, please let me know. That would be a lot faster than doing reordering the polynomial manually, particularly if it is in a factored form.

Thanks again for welcoming me into the Derive Users Group.

Cordially, Roger Folsom

BTW, it is not easy to find a meaningful translation of "Comparative Statics" into German, Josef  
Interesting readings on this issue can be found among others at:

[http://en.wikipedia.org/wiki/Comparative\\_statics](http://en.wikipedia.org/wiki/Comparative_statics)

<http://www.applet-magic.com/compstat.htm>

<http://www.economics.utoronto.ca/osborne/MathTutorial/DIFF.HTM>

<http://www.duke.edu/~dgraham/handouts/ComparativeStatics.pdf>



Some days ago I was phoned by Walter Klinger, a DERIVIAN from the first days, about an interesting behavior of DERIVE. It must be said that Walter has used several technologies in math teaching (Lower and Upper Secondary) during the last years and finally came back now to his “first love”, *DERIVE*. He explained the problem on the mobile phone and then sent a short *DERIVE* file to explain his problem more detailed:

## An Equation of Equations

Walter Klinger, Stockerau, Austria

Textbook 5th form (15 years), Example 899:

#1:  $[g := 5 \cdot x - 12 \cdot y = -3, h := 5 \cdot x - 12 \cdot y = 5, n := [5, -12]]$

#2:  $g = h$

??? What do we expect when solving this equation?

Input produced by students!

In fact there are two equations with tow unknowns!

#3:  $\text{SOLVE}(g = h, [x, y])$

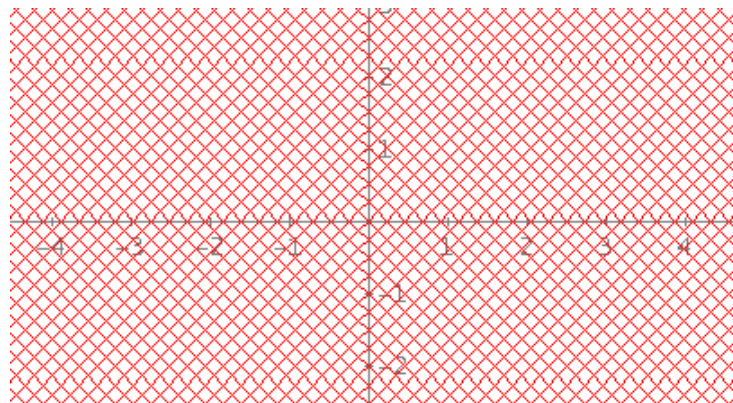
#4:  $5 \cdot x - 12 \cdot y \neq 5 \wedge 5 \cdot x - 12 \cdot y \neq -3$

??? Strange answer? How to interpret this answer?

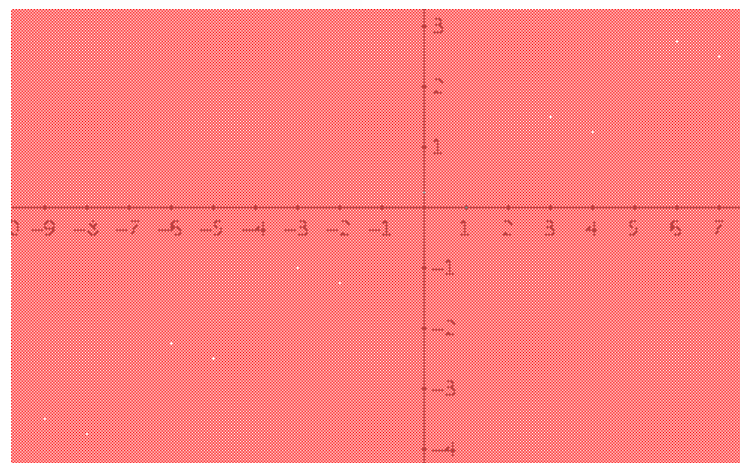
Lets plot it?

What is an equation of two equations?

Plot of expression #2



Plot of expression #4



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My first answer via phone was, that I cold imagine that *DERIVE* is interpreting h and g as statements and then treats expression  $h = g$  as a Boolean expression.

But I was not really sure and asked myself for an explanation for #4.

What to do? Simply ask *DERIVE* applying its really unique *Stepwise simplification*:

$$\#7: \quad g = h$$

$$\#8: \quad 5 \cdot x - 12 \cdot y = -3 = (5 \cdot x - 12 \cdot y = 5)$$

If p and q are of logical type,

$$p = q \rightarrow p \leftrightarrow q$$

$$\#9: \quad 5 \cdot x - 12 \cdot y = -3 \leftrightarrow 5 \cdot x - 12 \cdot y = 5$$

$$p \leftrightarrow q \rightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\#10: \quad ((5 \cdot x - 12 \cdot y \rightarrow 5 \cdot x - 12 \cdot y = 5) \wedge (5 \cdot x - 12 \cdot y = 5 \rightarrow 5 \cdot x - 12 \cdot y)) = (-3 \leftrightarrow 5 \cdot x - 12 \cdot y = 5)$$

$$p \rightarrow q \rightarrow \neg p \vee q$$

$$\#11: \quad ((\neg 5 \cdot x - 12 \cdot y \vee 5 \cdot x - 12 \cdot y = 5) \wedge (5 \cdot x - 12 \cdot y = 5 \vee 5 \cdot x - 12 \cdot y)) = (-3 \leftrightarrow 5 \cdot x - 12 \cdot y = 5)$$

The first steps confirmed my conjecture about treating  $g = h$  as a Boolean expression. Some steps are following until coming to an end which is given in #16:

If n and m are integers,

$$n \wedge m \rightarrow \text{bitwise\_logical\_and\_of\_n\_and\_m}$$

$$\#16: \quad ((5 \cdot x - 12 \cdot y = 5 \vee \neg 5 \cdot x - 12 \cdot y) \wedge (5 \cdot x - 12 \cdot y \neq 5 \vee 5 \cdot x - 12 \cdot y)) = ((5 \cdot x - 12 \cdot y = 5 \vee 2) \wedge (5 \cdot x - 12 \cdot y \neq 5 \vee -3))$$

Repeating ...

This would be the correct form of the Boolean expression

$$\#17: \quad g \wedge h$$

which gives the correct truth value for the contradiction

$$\#18: \quad \text{false}$$

My explanation for #4 from above

$$\#19: \quad 5 \cdot x - 12 \cdot y \neq 5 \wedge 5 \cdot x - 12 \cdot y \neq -3$$

The expression  $g = h$  or better  $g \leftrightarrow h$  is true if  $5 \cdot x - 12 \cdot y \neq 5$  AND  $5 \cdot x - 12 \cdot y \neq -3$   
i.e. it is false for  $5x - 12 = 5$  OR  $5x - 12y = -3$ .

Hence the whole plane is shaded (true) with exeption of (only some) the points on the lines g and h.  
Why only some points? This could be a deficiency of the shading algorithm??

I tried with two other equations of parallel lines and found out that the “non-plotting” of the “forbidden” points seems in fact to depend on the resolution of the screen and connected with this on the scaling of the plane.

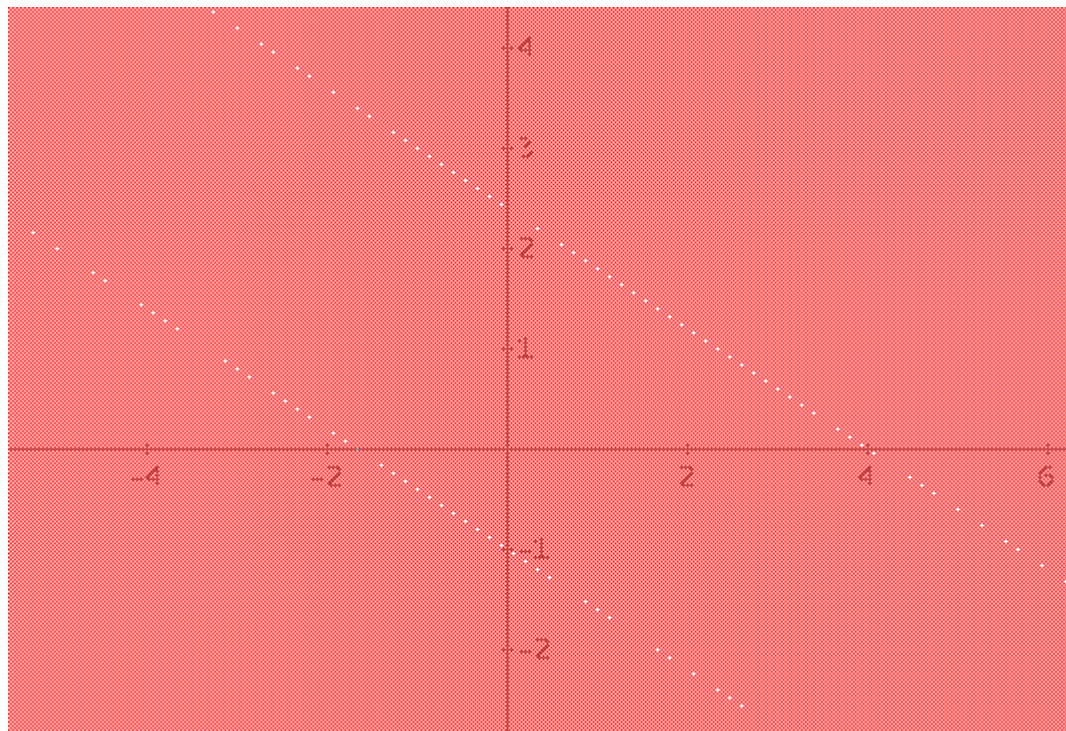
See the next example:

#1:  $[h := 3 \cdot x + 5 \cdot y = 12, g := 3 \cdot x + 5 \cdot y = -5, j := 2 \cdot x + 5 \cdot y = 5]$

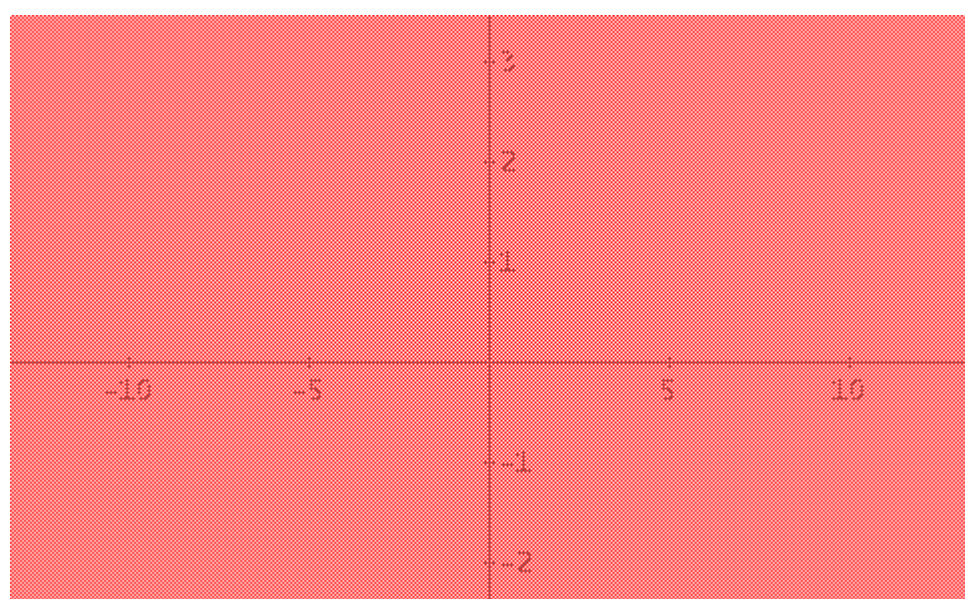
#2:  $\text{SOLVE}(h = g)$

#3:  $3 \cdot x + 5 \cdot y \neq 12 \wedge 3 \cdot x + 5 \cdot y \neq -5$

Many white spots within the shaded plane

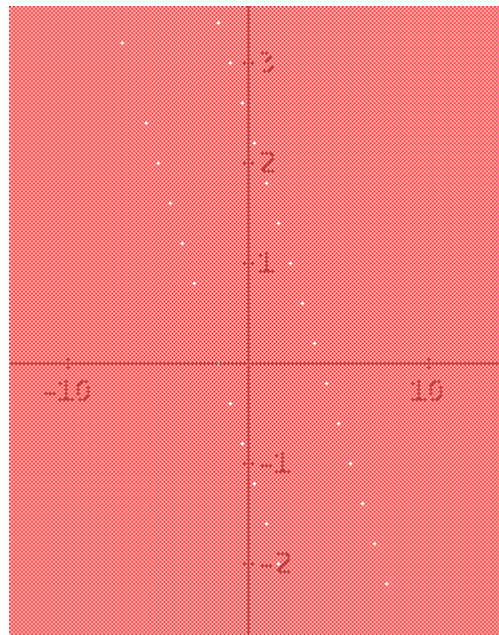


I change the scaling and there are no spots at all!



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There are the spots again but not the same as above.



My next question: What will happen in case of a solution? Take equations h and j.

#4: SOLVE(h = j)

#5:  $(3 \cdot x + 5 \cdot y = 12 \vee 2 \cdot x + 5 \cdot y \neq 5) \wedge (2 \cdot x + 5 \cdot y = 5 \vee 3 \cdot x + 5 \cdot y \neq 12)$

The (h,j) system has a unique solution,  
but ... #7 cannot be plotted

#6: SOLVE(h = j, x)

#7:  $\left( x = \frac{12 - 5 \cdot y}{3} \wedge x = \frac{5 \cdot (1 - y)}{2} \right) \vee \left( x \neq \frac{12 - 5 \cdot y}{3} \wedge x \neq \frac{5 \cdot (1 - y)}{2} \right)$

#8: SOLVE(h = h)

#9: true

Any comments on this interesting behavior are welcome. I tried other Computer Algebra Systems. Most of them do not accept the “equation of equations”.

I tried with changed resolution of the screen ( $1024 \times 768$  instead of  $1280 \times 1024$ ) and received other patterns in the shaded plane.