

THE BULLETIN OF THE



USER GROUP

+ CAS-TI

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During the last three months I came across some very useful and recommendable websites:

MATH e-NEWSLETTERS

Visit Prof. Michael de Villiers homepage and download the latest MATH e-NEWSLETTERS. They are full of links to interesting papers and other websites of interest.

<http://dynamicmathematicslearning.com/homepage4.html>

(From his last Newsletter: “*The three dimensions of a credit card are length, width and debt.*”)

Geometry Expressions Newsletters

can be downloaded together with lots of materials from

<http://geometryexpressions.com/explore.php?p=04-Newsletters>



Eberhard Lehmann has also a great webpage (in German only):

<http://home.snafu.de/mirza/>

I found a website filled with links to 83 (!!!) online books which can be downloaded for free.

<http://people.math.gatech.edu/~cain/textbooks/onlinebooks.html>

The concept of this site is given as follows:

The writing of textbooks and making them freely available on the web is an idea whose time has arrived. Most college mathematics textbooks attempt to be all things to all people and, as a result, are much too big and expensive. This perhaps made some sense when these books were rather expensive to produce and distribute--but this time has passed.

Examples are:

Book #1:	Multivariate Calculus
Book #45:	Difference Equations to Differential Equations
Book #83:	Topology Atlas

And there is another atlas on

<http://geometryatlas.com/>

An *Introduction to Tensor Calculus and Continuum Mechanics* can be downloaded from

<http://www.math.odu.edu/~jhh/bookpdf.zip>

Interesting facts about *Trochoids a.o.* can be found at

<http://www.math.odu.edu/~jhh/counter3.html>

Dear DUG Members,

I apologize for the delay releasing DNL#92. When I planned this issue I intended to publish several articles. I wanted to continue Dietmar Oertel's prime number investigations. Then there was a ready made article on Cilfford Pickover's *Mygalomorphs* spider nets. I generated these great 2D- and 3D-graphs with DERIVE. In the last German TI-News was an article, written by Alfred Roulier, Switzerland, presenting a Nspire-program written in LUA. I contacted him and he added very valuable comments and programs so that I (we) could realize these spider nets on the TI-Nspire.

When I received David Halprin's brainteaser (page 20) I remembered that he had promised in an earlier communication to update a contribution *Recursive Series*. I reminded David and he sent the requested article within a few days, many thanks.

But - as you will see - the article about *Solving Polynomial Systems*, which was a lecture given by our Canadian friends at ACA 2013 became very extended. It is the paper form of a ppt-presentation. Although this article deals mainly with TI-NspireCAS it is also of high interest for DERIVERS and for all CAS-users who want to get some insight

what happens behind. When I collected all materials suitable for publishing in the USER FORUM I noticed that I have to leave most of the intended - and ready made - articles for 2014.

We can be proud that the DUG has grown again in 2013. One of our new members is from IRAN. I didn't want to miss the opportunity including one of the articles which came in from Iran this fall. (page 38)

Please notice the many URLs presented on the information page. The websites offer numerous resources (more links, papers, and many online textbooks, all of them for free).

Finally you can find our wishes for 2014 on the last page of this DNL. Many thanks to Dietmar Oertel for "math-painting" this wonderful candle.

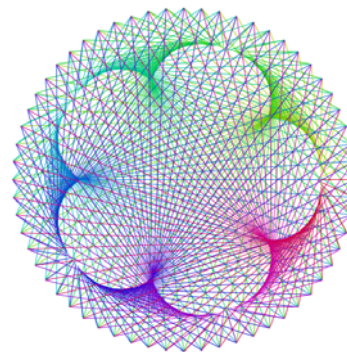
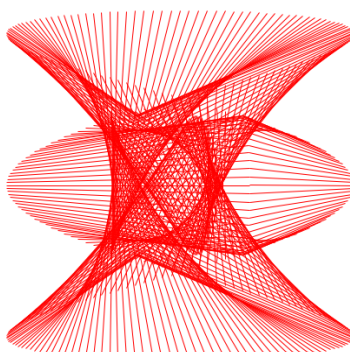
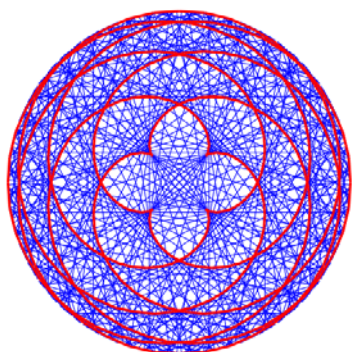
Best regards for 2014, the year of TIME 2014 (see also page 19).



I have to apologize but there is not even space for my letter in German, Josef

Download all DNL-DERIVE- and TI-files from

<http://www.austromath.at/dug/>



Some LUA – created Spider Nets

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue: March 2014

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Hill-Encryption, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland's Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what's new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips, USA
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Treating Differential Equations (M. Beaudin, G. Piccard, Ch. Trottier), CAN
A New Approach to Taylor Series, D. Oertel, GER
Cesar Multiplication, G. Schödl, AUT
Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT
Rational Hooks, J. Lechner, AUT
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT
Space Curves with adjustable Curvature and Torsion, P. Trebisz, GER
Recursive Series, D. Halprin, AUS
and others

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ACA 2013

Applications of Computer Algebra
Session: Computer Algebra in Education
Malaga, Spain, July 2-6

Solving Polynomial Systems with Nspire CAS

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Abstract

Nspire CAS can solve polynomial systems, using the Gröbner/Buchberger elimination method, but users don't have access to an explicit function like the one found in *Derive* (the so called "GROEBNER_BASIS" function). Thus we cannot see how the method is used when solving polynomial system with Nspire CAS. In this talk, we will show typical examples of polynomial systems that arise when teaching Lagrange multipliers technique. In the first part, the example will emphasise the importance of checking solutions and examining graphically the problem. The second example, in part two, will show a classic optimization problem where we will analyze the answer given by the commands "solve" and "zeros": we will find one wrong solution and some solutions will be missing (but simple parametric equations of the constraint will help us find the right answer). Using *Derive*'s GROEBNER_BASIS function, we will try to show what can yield this problem.

When teaching row reduced echelon form to students, we tell them that this is the way a linear system should be solved in general – instead of constantly applying the (black box) "SOLVE" command. In case of polynomial systems, access to a "Gröbner basis function" would be, for users, an important tool for understanding results obtained by the Nspire CAS system.

Keywords: Polynomial systems, solving facilities, Lagrange multipliers, Gröbner basis.

Overview

- Introduction
- Solving Polynomial Systems
 - Example 1: Everything is OK but...
 - Example 2: Something goes wrong.
- Gröbner Bases for Example 2
- Conclusion

Introduction

Each of our students (future engineers) has a TI-Nspire CX CAS handheld calculator on his desk. So, the “temptation” for using the “solve” or “zeros” command is great.

Our students are using one of these commands every time they need to solve an equation or a system of equations. We don’t want to restrict its use. Instead, we would like to guide its use. Very rarely, they are told to verify their answers (even for a single equation). In the case of a single equation in one variable, teachers should ask them to check *graphically* the answer they have found.

For a *linear* system of equations, the Gauss-Jordan method works perfectly and yields all solutions.

Students are still learning how to perform the row operations by hand. But they rapidly make use of the “rref” command (row reduced echelon form). At the end, they (*only*) need to be able to write down the answer in parametric form (in the case of infinity of solutions).

For a *polynomial* system of equations, if no initial guesses are specified, Nspire CAS uses the lexical Gröbner/Buchberger elimination method to attempt to determine all solutions. It would be fine to have access to such a function.

As we said before, we will present two examples.

The first one will look “complicated” because we will need to solve five equations. But Nspire CAS will easily solve the system, in exact mode, and get the right solution.

So everything will be OK. But the example will force us to question the answers given by a CAS system.

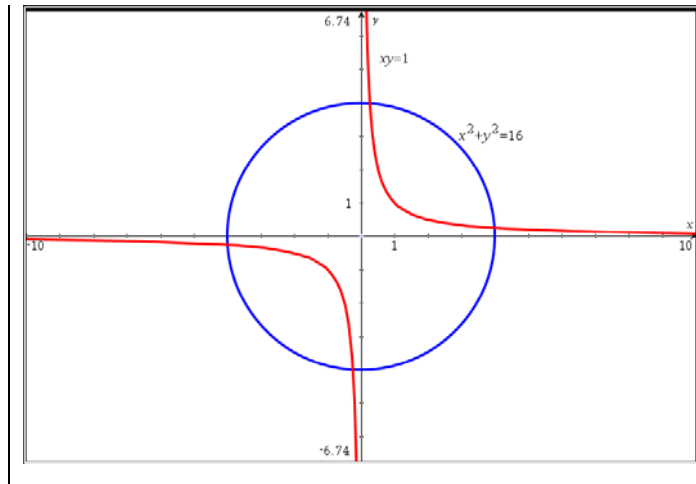
The second example will consist of three equations: a surprise will appear...

Let us mention that the examples we will show are a direct consequence of teaching mathematics with technology. Each of the two examples is interesting without technology. But with technology, the interest is greater.

What follows is a brief explanation about the theory of Gröbner bases:

Let's take a simple example. Suppose one needs to solve the following non linear (but polynomial) system:

$$\begin{cases} x^2 + y^2 = 16 \\ xy = 1 \end{cases}$$



We are in the case of a finite number of solutions. The idea: replace the original system by an equivalent one. With the following property: the first equation is now univariate. Selecting an order for the variable, one can obtain

$$\begin{cases} x^4 - 16x^2 + 1 = 0 \\ x^3 + y - 16x = 0 \end{cases}$$

Here is how to do it:

$$\begin{aligned} x^2 + y^2 - 16 &\rightarrow x^2(x^2 + y^2 - 16) \\ &= x^4 + (xy)^2 - 16x^2 = x^4 - 16x^2 + 1 \\ x^2 + y^2 - 16 &\rightarrow x(x^2 + y^2 - 16) \\ &= x^3 + xy^2 - 16x = x^3 + y(xy) - 16x \\ &= x^3 + y - 16x \end{aligned}$$

We first solve the equation $x^4 - 16x^2 + 1 = 0$ and then we substitute each value of x into $y + x^3 - 16x = 0$ to find the corresponding values of y .

More details:

Gröbner Bases: A Short Introduction for Systems Theorists by Bruno Buchberger.

<http://people.reed.edu/~davidp/pcmi/buchberger.pdf>

Solving Polynomial Systems: Example 1

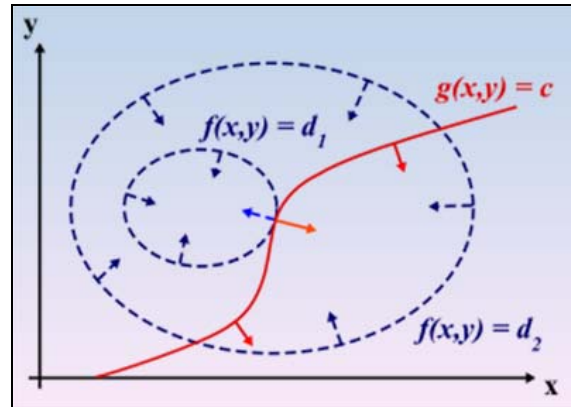
The problem: the plane $z = 1 + x + y$ intersects the cone $z^2 = x^2 + y^2$.

Find the points on the curve of intersection that are the closest and the farthest from the origin.

Possible approach: apply Lagrange multipliers technique.

Recall about Lagrange's method.

Here we have two constraints. In the case of one constraint, here is the idea. Suppose you need to find the extreme values of $f(x, y)$ under the constraint $g(x, y) = c$.



Now suppose we want to find the extreme values of $f(x, y, z)$ subject to two constraints $g(x, y, z) = 0$, $h(x, y, z) = 0$. Let $P = (x, y, z)$ be a solution point.

If such a point P exists, then (assuming that the gradient vectors are not zero and not parallel) there exists numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$$

Taking in account the two constraints, we need to solve the following system:

$$\begin{cases} \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} - \mu \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} - \mu \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} - \lambda \frac{\partial g}{\partial z} - \mu \frac{\partial h}{\partial z} \\ g \\ h \end{cases} \quad \begin{pmatrix} \nabla f(P) - \lambda \nabla g(P) - \mu \nabla h(P) = \mathbf{0} \\ g = 0 \\ h = 0 \end{pmatrix}$$

In our example let

$f := x^2 + y^2 + z^2$	(square of the distance to the origin)
$co1 := z - 1 - x - y$	($co1 = 0$ is the first surface: the plane)
$co2 := z^2 - x^2 - y^2$	($co2 = 0$ is the second surface: the cone)

Note that f is the expression we want to optimize; $co1 = 0$ is the first constraint and $co2 = 0$ is the second constraint.

We need to find the (real) zeros of the following system:

$$\nabla f - \lambda \nabla co1 - \mu \nabla co2 = 0$$

$$co1 = 0$$

$$co2 = 0$$

This yields the system:

$$syst1 := \begin{cases} 2(\mu+1)x + \lambda, 2(\mu+1)y + \lambda, -2(\mu-1)z - \lambda, \\ -x - y + z - 1, -x^2 - y^2 + z^2 \end{cases}$$

Here is the Nspire solution (without looking at the graphs):. We have used Lagrange multipliers technique, solving five equations. Two solutions are obtained by the CAS:

$f := (\text{norm}([x \ y \ z]))^2$	Expression we want to optimize	$x^2 + y^2 + z^2$
$co1 := z - 1 - x - y; co2 := z^2 - x^2 - y^2$	The two constraints	$-x^2 - y^2 + z^2$
$\left\{ \frac{d}{dx}(f) - \lambda \cdot \frac{d}{dx}(co1) - \mu \cdot \frac{d}{dx}(co2), \frac{d}{dy}(f) - \lambda \cdot \frac{d}{dy}(co1) - \mu \cdot \frac{d}{dy}(co2), \frac{d}{dz}(f) - \lambda \cdot \frac{d}{dz}(co1) - \mu \cdot \frac{d}{dz}(co2), co1, co2 \right\}$		
Applying Lagrange's method	$\{(2 \cdot \mu + 2) \cdot x + \lambda, (2 \cdot \mu + 2) \cdot y + \lambda, (2 - 2 \cdot \mu) \cdot z - \lambda, -x - y + z - 1, -x^2 - y^2 + z^2\}$	
$syst1 := \{(2 \cdot \mu + 2) \cdot x + \lambda, (2 \cdot \mu + 2) \cdot y + \lambda, (2 - 2 \cdot \mu) \cdot z - \lambda, -x - y + z - 1, -x^2 - y^2 + z^2\}$		
$syst1$ is the polynomial system we need to solve	$\{(2 \cdot \mu + 2) \cdot x + \lambda, (2 \cdot \mu + 2) \cdot y + \lambda, (2 - 2 \cdot \mu) \cdot z - \lambda, -x - y + z - 1, -x^2 - y^2 + z^2\}$	
$mat1 := \text{zeros}(syst1, \{x, y, z, \lambda, \mu\})$	$\begin{bmatrix} \frac{\sqrt{2}-2}{2} & \frac{\sqrt{2}-2}{2} & \sqrt{2}-1 & -4 \cdot (2 \cdot \sqrt{2}-3) & -(2 \cdot \sqrt{2}-3) \\ -(\frac{\sqrt{2}+2}{2}) & -(\frac{\sqrt{2}+2}{2}) & -(\sqrt{2}+1) & 4 \cdot (2 \cdot \sqrt{2}+3) & 2 \cdot \sqrt{2}+3 \end{bmatrix}$	
Solutions of the system		
$p1 := \text{subMat}(mat1, 1, 1, 1, 3)$	We extract x, y and z from each row of the matrix $mat1$.	$\begin{bmatrix} \frac{\sqrt{2}-2}{2} & \frac{\sqrt{2}-2}{2} & \sqrt{2}-1 \end{bmatrix}$
$p2 := \text{subMat}(mat1, 2, 1, 2, 3)$	We have 2 points $p1$ and $p2$.	$\begin{bmatrix} -(\frac{\sqrt{2}+2}{2}) & -(\frac{\sqrt{2}+2}{2}) & -(\sqrt{2}+1) \end{bmatrix}$
$p1 \blacktriangleright$ Decimal		$[-0.292893 \ -0.292893 \ 0.414214]$
$p2 \blacktriangleright$ Decimal		$[-1.70711 \ -1.70711 \ -2.41421]$
$\text{norm}(p1)$		0.585786
$\text{norm}(p2)$		3.41421

Closest point seems to be $P1 = (-0.29, -0.29, 0.41)$

Farthest point seems to be $P2 = (-1.71, -1.71, -2.41)$

So Nspire CAS has found two solutions for system “ $syst1$ ” of five equations in five unknowns. These are, in fact, the two (real) solutions that exist.

The “zeros” command did a good job. But do we have a closest point and a farthest point?

In fact, P1 is the closest point from the origin. P2 is **NOT** the farthest point: it is a local minimum.

The “educational aspect” is the following: the method of Lagrange multipliers gives a *constrained extremum* only if one exists.

Let us say again: one point (P1) is the closest on to the origin. **There is NO farthest point from the origin!** You won't discover this if you don't push a little bit more this problem. For doing so, the CAS can (again) be very useful.

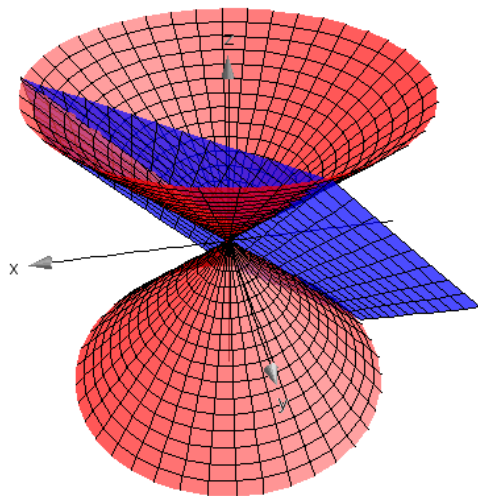
We can use the CAS to plot both surfaces in a first attempt. Then we can try to find parametric equations for the intersection curve.

If you give this kind of problem to your students, they need to know how to plot a surface with their CAS.

In order to plot the intersection curve they need to find – by themselves – parametric equations for the space curve. This curve is a hyperbola. On this curve, we can go as far as we want from the origin: there is no farthest point.

One thing is sure: without technology, it is difficult – but possible – to solve this example.

Here is what we can obtain: we have used NspireCAS (OS 3.2). The plane has been entered in function mode, the cone using parametric representation.



z1	(x,y) =	1+x+y
----	---------	-------

xp1	(t,u) =	t · cos(u)
yp1	(t,u) =	t · sin(u)
zp1	(t,u) =	t

Here are some parametric equations representing the curve of intersection:

$$\left\{ x=t, y=-\frac{2t+1}{2(t+1)}, z=t+\frac{1}{2(t+1)}; \quad t \in \mathbb{R}, t \neq -1 \right\}$$

(Students can find them by using a substitution of the z-value of the plane into the cone equation.

In details:

$$(1+x+y)^2 = x^2 + y^2 \Leftrightarrow 2x(y+1) + 2y+1 = 0$$

$$\text{Solve}(2x(y+1) + 2y+1 = 0, y) \rightarrow y = -\frac{2x+1}{2(x+1)}$$

Another way to see this:

$$\text{GROEBNER_BASIS}\left([z - 1 - x - y, z^2 - x^2 - y^2], [z, x, y]\right)$$

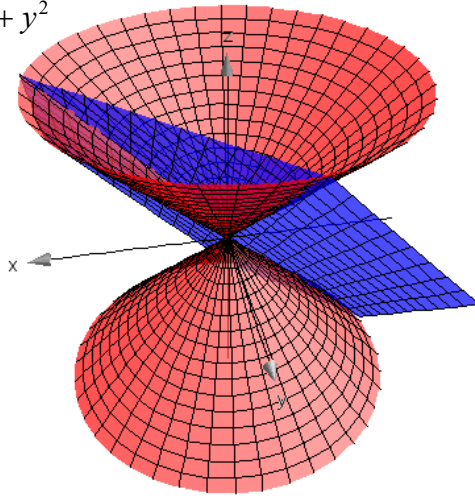
$$[x \cdot (2 \cdot y + 2) + 2 \cdot y + 1, z - x - y - 1]$$

In conclusion, here are two surfaces, the curve of intersection and the two points:

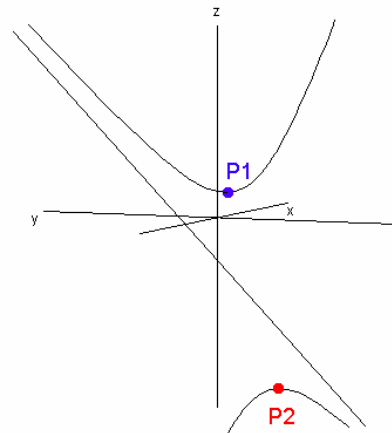
P1 = (-0.29, -0.29, 0.41) – the closest one – and P2 = (-1.71, -1.71, -2.41).

$$z = 1 + x + y$$

$$z^2 = x^2 + y^2$$



$$\left\{ x = t, y = -\frac{2t+1}{2(t+1)}, z = t + \frac{1}{2(t+1)}; \quad t \in \mathbb{R}, t \neq -1 \right\}$$



Final remark about example 1: technology allows us to link single and multiple variable calculus.

Once parametric equations have been found, we can compute the norm of the vector position and find its minimum value again.

The following Nspire CAS screens show this.

mat1		$\begin{bmatrix} -0.292893 & -0.292893 & 0.414214 & 0.686292 & 0.171573 \\ -1.70711 & -1.70711 & -2.41421 & 23.3137 & 5.82843 \end{bmatrix}$
solve($co1=0,z$)		$z=x+y+1$
co2 $ z=x+y+1$		$2 \cdot x \cdot (y+1) + 2 \cdot y + 1$
zeros($2 \cdot x \cdot (y+1) + 2 \cdot y + 1, y$)		$\left\{ \frac{-(2 \cdot x + 1)}{2 \cdot (x + 1)} \right\}$
$1 + x + \frac{-(2 \cdot x + 1)}{2 \cdot (x + 1)}$		$\frac{1}{2 \cdot (x + 1)} + x$
$rr := \left[x \cdot \frac{-(2 \cdot x + 1)}{2 \cdot (x + 1)}, \frac{1}{2 \cdot (x + 1)} + x \right]$		$\left[x \cdot \frac{-(2 \cdot x + 1)}{2 \cdot (x + 1)}, \frac{1}{2 \cdot (x + 1)} + x \right]$
norm(rr)		$\frac{\sqrt{2} \cdot (2 \cdot x^2 + 2 \cdot x + 1)}{2 \cdot x + 1 }$
fMin($\frac{\sqrt{2} \cdot (2 \cdot x^2 + 2 \cdot x + 1)}{2 \cdot x + 1 }, x$)		$x = \frac{\sqrt{2} - 2}{2}$
$rr x = \frac{\sqrt{2} - 2}{2}$		$\begin{bmatrix} -0.292893 & -0.292893 & 0.414214 \end{bmatrix}$ This is P1
$\text{norm}(rr) _{x = \frac{\sqrt{2} - 2}{2}}$		$2 - \sqrt{2}$
$(2 - \sqrt{2}) \blacktriangleright \text{Decimal}$		0.585786

Solving Polynomial Systems: Example 2

As long as the “zeros” (or “solve”) command gives the user correct answers, there is no motivation to try to understand which algorithm the “solve” command is using when solving polynomial systems.

This algorithm is “Gröbner Bases” and Nspire CAS is using it. But users don’t have access to a “Gröbner basis function”.

It is true that this stuff is not an undergraduate subject but a “Gröbner basis function” would play a role like the “rref” command plays for linear systems.

Consequently, we would like Texas Instruments to make this function available for the user.

Example 2 will show that such a function would be useful.

Recall from calculus: when we want to find the absolute (global) maximum value of a function f of two variables over a closed and bounded domain D , we need to consider the interior of D and its boundary.

Because a continuous function defined over a compact set (closed and bounded in this case) reaches a global maximum value, such a point exists. It can be inside the domain or on the boundary.

We first need to find the critical points located inside the domain D .

The second derivative test for functions of 2 variables can be used to classify these points.

And Lagrange multiplier can be used for the boundary.

And if the boundary consists of a circle (as in the next example), parametric equations can be used instead of Lagrange’s method.

This will be our first attempt to locate the maximum value on the boundary.

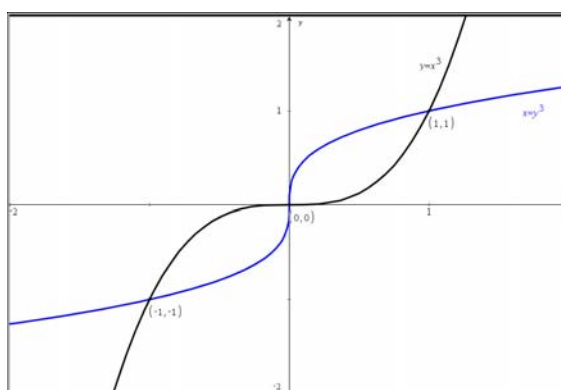
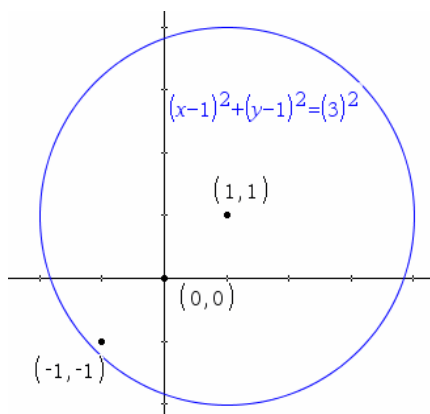
This way, we will already know what is the absolute maximum value of our function. Lagrange’s method should eventually confirm this...

Here is the example.

The temperature at a point (x, y) in the plane is given by $te(x, y) = x^4 + y^4 - 4xy + 2$.

We want to find the location of the hottest point on the disk $(x-1)^2 + (y-1)^2 \leq 9$.

Three critical points are located *inside* the disk: $(0, 0)$, $(1, 1)$ and $(-1, -1)$.



$$\nabla te(x, y) = [4x^3 - 4y, 4y^3 - 4x] = [0, 0] \Leftrightarrow \begin{cases} x = 0 \text{ and } y = 0 \\ x = 1 \text{ and } y = 1 \\ x = -1 \text{ and } y = -1 \end{cases}$$

The points (1, 1) and (-1, -1) are local minimums where both temperatures are 0. And (0, 0) is a saddle point where the temperature value is 2.

This is a consequence of the second derivative test for local extreme values.

But we don't need to recall this. We want to find the *maximum* temperature, the point (2, 2) is inside the disk and

$$te(x, y) = x^4 + y^4 - 4xy + 2 \Rightarrow te(2, 2) = 18.$$

So the absolute maximum is achieved on the circular boundary $(x-1)^2 + (y-1)^2 = 9$.

As said before, we won't start by using Lagrange's method. Instead, let's use parametric equations for the circle.

$$(x-1)^2 + (y-1)^2 = 9 \Leftrightarrow \left(\frac{x-1}{3}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1.$$

We can use the first trig identity: $\sin^2 t + \cos^2 t = 1$.

Using $x = 1 + 3 \cos(t)$ and $y = 1 + 3 \sin(t)$ where t is the parameter ($-p \leq t \leq p$), we can plot the graph of the temperature as a function of t .

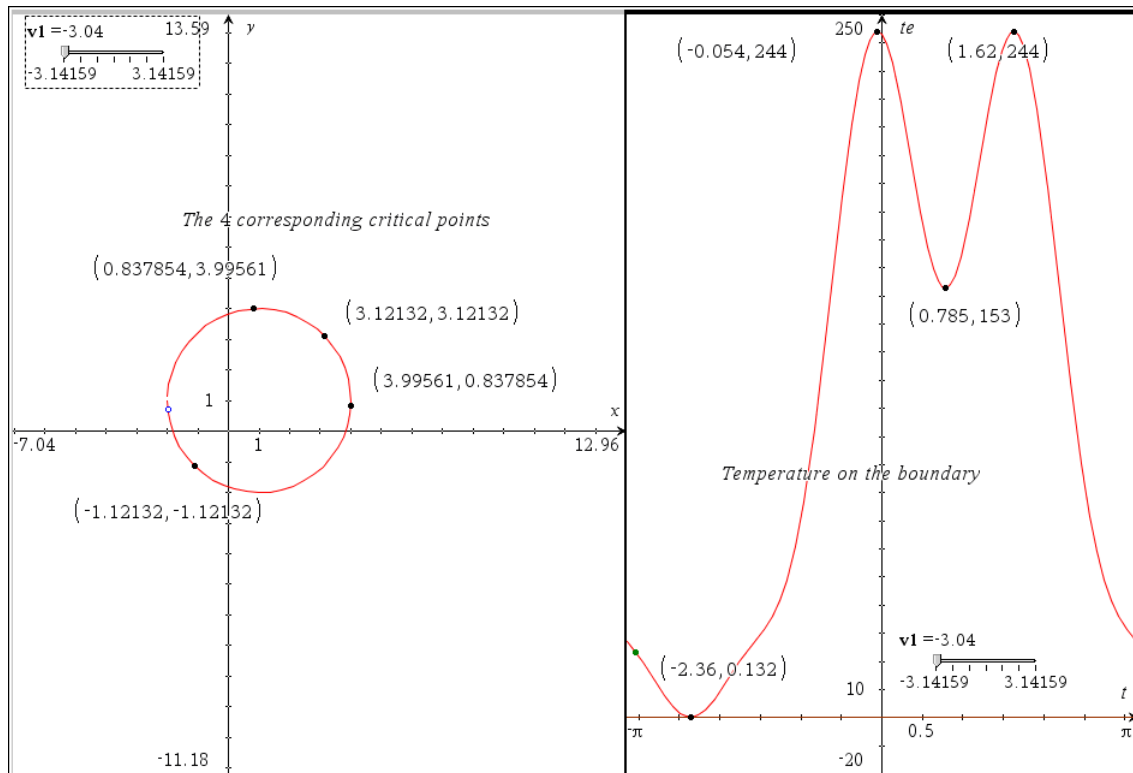
We will show that the **maximum (temperature) value is 243.98** and is achieved twice:

when $t = -0.0541$, the corresponding point on the circle
being (3.996, 0.8379);

when $t = 1.625$, the corresponding point on the circle
being (0.8379, 3.996).

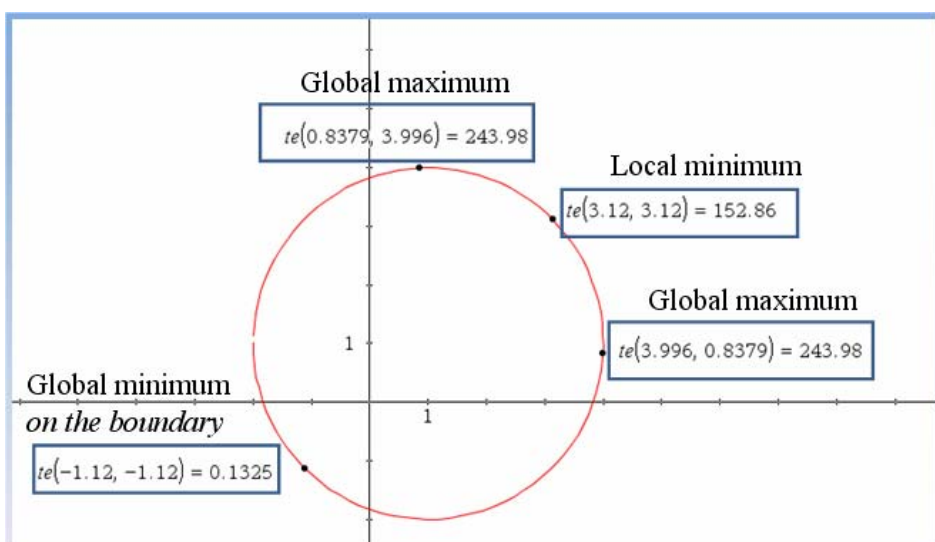
We will see that the minimum value on the boundary is 0.1325 and is achieved at the value $t = -2.356$. The corresponding point on the circle is (-1.12, -1.12).

There is also a local minimum of 152.868 at $t = 0.7854$; the point on the circle is (3.12, 3.12).



- So, parametric equations has transformed this *multiple* variable calculus problem into a *single* variable problem.

Value of t	Corresponding point	Value of the temperature
- 2.356	(- 1.12, - 1.12)	0.1325
- 0.0541	(3.996, 0.8379)	243.98
0.7854	(- 3.12, - 3.12)	152.86
1.625	(0.8379, 3.996)	243.98



Using Lagrange's method with level curves, this should allow us to locate 4 solutions. Let's see this using *Derive* (we need a fast implicit plotter here).

We will plot the constraint, the 4 points on it and some level curves of the temperature function.

At each point, we should observe that one gradient vector is a multiple of the other.

Now let's use Lagrange's method algebraically. We want to optimize the function $te(x, y)$ under the constraint g where

$$\begin{aligned} te(x, y) &= x^4 + y^4 - 4xy + 2, \\ g &= (x-1)^2 + (y-1)^2 - 9 = 0. \end{aligned}$$

Lagrange's method (one constraint) tells us to solve the system

$$\begin{cases} \nabla te = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow \begin{cases} 4x^3 - 2\lambda x - 4y + 2\lambda = 0 \\ -4x + 4y^3 - 2\lambda y + 2\lambda = 0 \\ x^2 - 3x + y^2 - 2y - 7 = 0 \end{cases}$$

Here is the **surprise**: Nspire CAS finds 3 solutions (not 4) and one is wrong (does not satisfy the constraint). The 2 others are good but are minimums (the local one and the global one as seen before).

So, with Lagrange, Nspire CAS *does not* find the absolute maximum!

Strange and important thing: the wrong solution is given numerically and the 2 good ones are in exact mode.

The following screen shows all this.

The screenshot shows the Nspire CAS interface with the following content:

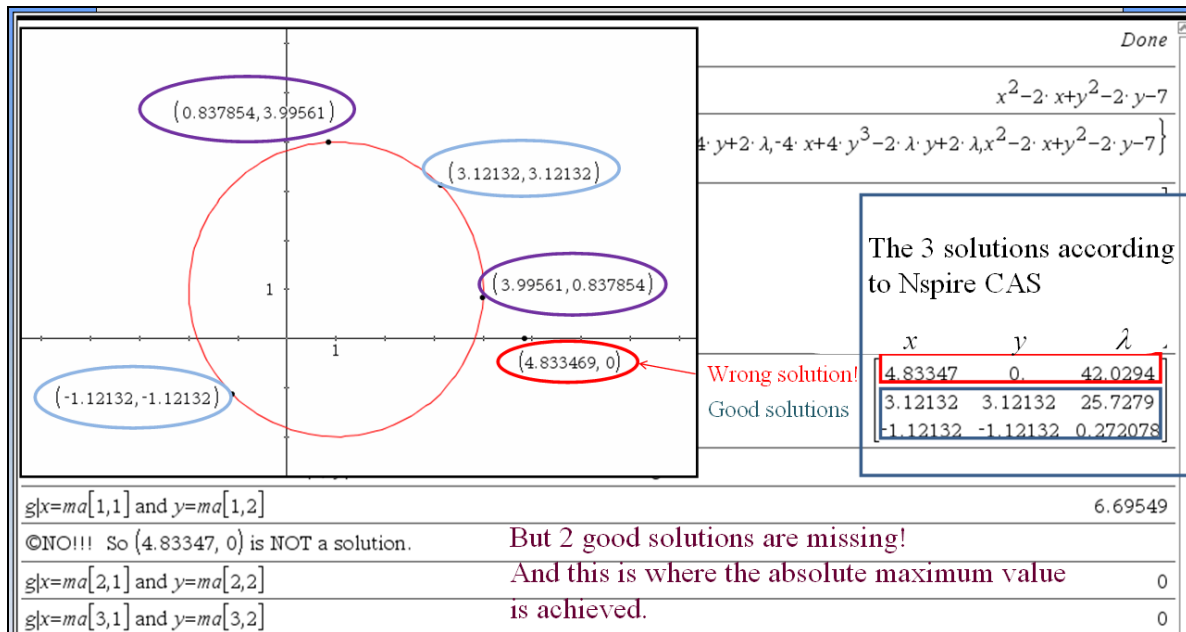
- Define functions:**
 - $te(x,y) := x^4 + y^4 - 4 \cdot x \cdot y + 2$
 - $g := (x-1)^2 + (y-1)^2 - 9$
- Solve the system:**

$$\text{syst} := \left\{ \frac{d}{dx}(te(x,y)) - \lambda \cdot \frac{d}{dx}(g), \frac{d}{dy}(te(x,y)) - \lambda \cdot \frac{d}{dy}(g), g \right\}$$
- Find solutions:**

$$ma := \text{zeros}(\text{syst}, \{x, y, \lambda\})$$
- Display solutions:**

ma ▶ Decimal

x	y	λ
4.83347	0.	42.0294
3.12132	3.12132	25.7279
-1.12132	-1.12132	0.272078
- Verification:**
 - Let us check if each solution (x, y) in "ma" satisfies the constraint "g".
 - $g|x=ma[1,1]$ and $y=ma[1,2]$ → 6.69549
 - ⊙NO!!! So (4.83347, 0) is NOT a solution.
 - $g|x=ma[2,1]$ and $y=ma[2,2]$ → 0
 - $g|x=ma[3,1]$ and $y=ma[3,2]$ → 0
 - ⊙Yes, the 2 other solutions are good. The (numerical) value of "te" at each of these 2 points is:
 - $te(ma[2,1], ma[2,2])$ → 152.868
 - $te(ma[3,1], ma[3,2])$ → 0.132468



Gröbner Bases for Example 2

In order to see what went wrong, we will solve the former system, using a Gröbner basis provided by *Derive*.

Some interesting mathematical aspects will be revealed.

Let us recall that the system we need to solve is the following (we have copied the system from Nspire CAS to *Derive*).

$$\text{syst} := \left[4 \cdot x^3 - 2 \cdot \lambda \cdot x - 4 \cdot y + 2 \cdot \lambda, -4 \cdot x + 4 \cdot y^3 - 2 \cdot \lambda \cdot y + 2 \cdot \lambda, x^2 - 2 \cdot x + y^2 - 2 \cdot y - 7 \right]$$

Important to say: in exact mode, only 2 answers will be shown by *Derive*.

Increasing the precision and using approximate mode, 4 answers will come out:

#1: $\text{syst} := \left[4 \cdot x^3 - 2 \cdot \lambda \cdot x - 4 \cdot y + 2 \cdot \lambda, -4 \cdot x + 4 \cdot y^3 - 2 \cdot \lambda \cdot y + 2 \cdot \lambda, x^2 - 2 \cdot x + y^2 - 2 \cdot y - 7 \right]$

#2: $\text{SOLUTIONS}(\text{syst}, [x, y, \lambda], \text{Real})$

#3:
$$\begin{bmatrix} \frac{3 \cdot \sqrt{2}}{2} + 1 & \frac{3 \cdot \sqrt{2}}{2} + 1 & 9 \cdot \sqrt{2} + 13 \\ 1 - \frac{3 \cdot \sqrt{2}}{2} & 1 - \frac{3 \cdot \sqrt{2}}{2} & 13 - 9 \cdot \sqrt{2} \end{bmatrix}$$

#4:
$$\text{APPROX} \begin{bmatrix} \frac{3 \cdot \sqrt{2}}{2} + 1 & \frac{3 \cdot \sqrt{2}}{2} + 1 & 9 \cdot \sqrt{2} + 13 \\ 1 - \frac{3 \cdot \sqrt{2}}{2} & 1 - \frac{3 \cdot \sqrt{2}}{2} & 13 - 9 \cdot \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3.121320343 & 3.121320343 & 25.72792206 \\ -1.121320343 & -1.121320343 & 0.2720779386 \end{bmatrix}$$

#5: $\text{APPROX}(\text{SOLUTIONS}(\text{syst}, [x, y, \lambda], \text{Real}), 30)$

#6:
$$\begin{bmatrix} 3.99561492055584386304002845492 & 0.837854239206801150904033446716 & 42.0293624379018996177972474758 \\ 3.12132034355964257320253308633 & 3.12132034355964257320253308631 & 25.7279220613578554392151985178 \\ -1.12132034355964257320253308631 & -1.12132034355964257320253308631 & 0.272077938642144560784801482114 \\ 0.837854239206801150904033452694 & 3.99561492055584386304002845059 & 42.0293624379018996177972474758 \end{bmatrix}$$

Remember the good Nspire-solutions (within the red box):

4.83347	0	42.0294
$\frac{3 \cdot \sqrt{2} + 2}{2}$	$\frac{3 \cdot \sqrt{2} + 2}{2}$	$9 \cdot \sqrt{2} + 13$
$-\frac{(3 \cdot \sqrt{2} - 2)}{2}$	$-\frac{(3 \cdot \sqrt{2} - 2)}{2}$	$-(9 \cdot \sqrt{2} - 13)$

A Gröbner basis can help us to understand what caused the bug in Nspire CAS. If we use the lexicographic order l, x, y , the Gröbner basis consists of 3 elements.

$$\text{syst} := \left[4 \cdot x^3 - 2 \cdot \lambda \cdot x - 4 \cdot y + 2 \cdot \lambda, -4 \cdot x + 4 \cdot y^3 - 2 \cdot \lambda \cdot y + 2 \cdot \lambda, x^2 - 2 \cdot x + y^2 - 2 \cdot y - 7 \right]$$

$$\text{GROEBNER_BASIS}(\text{syst}, [\lambda, x, y])$$

$$\left[4 \cdot y^8 - 20 \cdot y^7 - 26 \cdot y^6 + 202 \cdot y^5 - 34 \cdot y^4 - 564 \cdot y^3 + 538 \cdot y^2 + 492 \cdot y - 511, 837 \cdot x - 48 \cdot y^7 + 244 \cdot y^6 + 240 \cdot y^5 - 2010 \cdot y^4 + 64 \cdot y^3 + 3952 \cdot y^2 - 2249 \cdot y - 3451, 7533 \cdot \lambda - 2152 \cdot y^7 + 9472 \cdot y^6 + 19068 \cdot y^5 - 93928 \cdot y^4 - 39456 \cdot y^3 + 247758 \cdot y^2 - 112822 \cdot y - 337036 \right]$$

This is *probably not* the Gröbner basis Nspire CAS used because it would have given the 4 solutions. Let's see this:

grob11(y)	$4 \cdot y^8 - 20 \cdot y^7 - 26 \cdot y^6 + 202 \cdot y^5 - 34 \cdot y^4 - 564 \cdot y^3 + 538 \cdot y^2 + 492 \cdot y - 511$
grob12(x, y)	$837 \cdot x - 48 \cdot y^7 + 244 \cdot y^6 + 240 \cdot y^5 - 2010 \cdot y^4 + 64 \cdot y^3 + 3952 \cdot y^2 - 2249 \cdot y - 3451$
grob13(λ, y)	$7533 \cdot \lambda - 2152 \cdot y^7 + 9472 \cdot y^6 + 19068 \cdot y^5 - 93928 \cdot y^4 - 39456 \cdot y^3 + 247758 \cdot y^2 - 112822 \cdot y - 337036$

Solving $\text{grob11}(y) = 0$ yields the 4 expected values of y , say $y1, y2, y3$ and $y4$. Then solving $\text{grob12}(x, y) = 0$ for each value of y yields the 4 expected values of x . Finally, for each of the 4 values of y , solving $\text{grob13}(y, l) = 0$ yields 3 (not 4) corresponding values of l . So the Gröbner basis "grob1" would have lead to the 4 correct answers:

x	y	λ
$\frac{-(3 \cdot \sqrt{2} - 2)}{2}$	$\frac{-(3 \cdot \sqrt{2} - 2)}{2}$	$-(9 \cdot \sqrt{2} - 13)$
3.99561492056	0.837854239207	42.0293624379
$\frac{3 \cdot \sqrt{2} + 2}{2}$	$\frac{3 \cdot \sqrt{2} + 2}{2}$	$9 \cdot \sqrt{2} + 13$
0.837854239484	3.99561492056	42.0293624398

Now, let us change the order of the lexical elimination method of Gröbner basis.

Let's use the order x, y, l . In this case, the Gröbner basis provided by *Derive* — we will call it "grob2"— consists of 4 elements. We will copy it to Nspire CAS. The first one ("grob21") is a polynomial of degree 5 in l . The next two are in (y, l) variables. The fourth contains x, y and l .

GROEBNER_BASIS(syst, [x, y, λ])

$$\left[\begin{aligned} &\lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364, y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) \\ &- \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364, 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + \\ &122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 128090340, 401898 \cdot x + \\ &401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634 \end{aligned} \right]$$

Derive

$$\text{grob2} := \left[\begin{aligned} &\lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364 \\ &y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364 \\ &2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 128090340 \\ &401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634 \end{aligned} \right]$$

$$\text{grob21}(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$$

Nspire CAS

$$\text{grob22}(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$$

$$\text{grob23}(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 128090340$$

$$\text{grob24}(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$$

Then we make the same analysis as before.

Nspire CAS finds the 3 real roots of $\text{grob21}(\lambda)$ because grob21 factors into a quadratic polynomial multiplied by a cubic polynomial (and $\lambda 3$ is the only real root of this cubic polynomial).

$\text{grob21}(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$	Done
$\text{grob22}(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$	Done
$\text{grob23}(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 128090340$	Done
$\text{grob24}(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$	Done
$\text{factor}(\text{grob21}(\lambda))$	$(\lambda^2 - 26 \cdot \lambda + 7) (\lambda^3 - 56 \cdot \lambda^2 + 636 \cdot \lambda - 2052)$
$\text{zeros}(\lambda^2 - 26 \cdot \lambda + 7)$	$\lambda 1, \lambda 2 \rightarrow \{-9 \cdot \sqrt{2} - 13, 9 \cdot \sqrt{2} + 13\}$
$\text{zeros}(\lambda^3 - 56 \cdot \lambda^2 + 636 \cdot \lambda - 2052)$	$\lambda 3 \rightarrow \{42.0294\}$
$\text{factor}(\text{grob22}(y, \lambda))$	$(\lambda^3 - 56 \cdot \lambda^2 + 636 \cdot \lambda - 2052) (6 \cdot y - \lambda + 7)$

The problem is when we substitute the value $\lambda 3$ into $\text{grob22}(y, \lambda)$: it should be identically 0 BECAUSE $\lambda 3$ is a root of the equation

$$\lambda^3 - 56 \lambda^2 + 636 \lambda - 2052 = 0$$

In Exact Mode – and this why *Derive* earlier returned only two solutions –, there is no way to find out that $\text{grob22}(y, \lambda) \equiv 0$ (nested radicals are causing problems to Computer Algebra Systems).

Derive

$$\text{grob21}(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$$

$$\text{grob22}(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$$

$$\text{grob23}(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 128090340$$

$$\text{grob24}(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$$

$$\text{SOLUTIONS}(\text{grob21}(\lambda), \lambda, \text{Real}) = \left[9 \cdot \sqrt{2} + 13, 13 - 9 \cdot \sqrt{2}, \frac{(43046 - 18 \cdot \sqrt{3561})^{1/3}}{3} + \frac{(18 \cdot \sqrt{3561} + 43046)^{1/3}}{3} + \frac{56}{3} \right]$$

$\lambda_3 := \frac{(43046 - 18 \cdot \sqrt{3561})^{1/3}}{3} + \frac{(18 \cdot \sqrt{3561} + 43046)^{1/3}}{3} + \frac{56}{3}$ This is $\lambda_3: \approx 42.029 \dots$

After the substitution of λ_3 into grob22

$$y \cdot (6 \cdot \lambda_3^3 - 336 \cdot \lambda_3^2 + 3816 \cdot \lambda_3 - 12312) - \lambda_3^4 + 63 \cdot \lambda_3^3 - 1028 \cdot \lambda_3^2 + 6504 \cdot \lambda_3 - 14364$$

#4 simplifies to $0 \cdot y + \varepsilon$ where ε is the constant in #6

$$\frac{4 \cdot (53848162016064514 - 89788612484790 \cdot \sqrt{3561})^{1/3}}{81} - \frac{4 \cdot (89788612484790 \cdot \sqrt{3561} + 53848162016064514)^{1/3}}{81} - \frac{2456 \cdot (231763985 - 193707)}{81}$$

$$\frac{2456 \cdot (193707 \cdot \sqrt{3561} + 231763985)^{1/3}}{81} + \frac{86092 \cdot (43046 - 18 \cdot \sqrt{3561})^{1/3}}{81} + \frac{86092 \cdot (18 \cdot \sqrt{3561} + 43046)^{1/3}}{81}$$

This is ε

$-3.213866225 \cdot 10^{-7}$ The constant ε in #6 is 0 but this can't be verified in exact mode. Increasing the precision to 20 digits, *Derive* approximates $\text{grob22}(y, \lambda_3)$ to 0:

Approximating ε , using 10 digits.

$\text{APPROX}(\text{grob22}(y, \lambda_3), 20) \rightarrow 0$

Also, note the following approximations of λ_3 while increasing the precision in *Derive*:

$$\lambda_3 := \frac{(43046 - 18 \cdot \sqrt{3561})^{1/3}}{3} + \frac{(18 \cdot \sqrt{3561} + 43046)^{1/3}}{3} + \frac{56}{3}$$

$\text{APPROX}(\lambda_3, 13) \rightarrow 42.02936243790$

$\text{APPROX}(\lambda_3, 15) \rightarrow 42.0293624379018$

Now let's switch to Nspire CAS and substitute λ_3 into grob22 . What happens to $\text{grob22}(y, \lambda_3)$? We know that this *should* be identically 0. In *Derive*, this was $0 \cdot y + \varepsilon$ with ε approximating to 0.

The fact that *Derive* was able to find $0 \cdot y + \varepsilon$ was crucial because NO value of y have been found using $\text{grob22}(y, \lambda_3)$. If, instead of $0 \cdot y + \varepsilon$, we take $\varepsilon_1 \cdot y + \varepsilon_2$, then a value of y can result from solving for y . And if $\varepsilon_2 = 0$, this value will be 0...!

On both the software version and the handheld, the computational accuracy of Nspire CAS is the following: floating-point (decimal) values in memory are stored using up to 14 digits with a 3-digit exponent. When a floating-point value is displayed, the displayed value is rounded as specified by the applicable mode settings *with a maximum of 12 digits*.

In the next Nspire CAS session (screen), we have set "Display Digits" to "Fix 12" for a better understanding.

$grob21(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$	Done
$grob22(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$	Done
$grob23(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 1280903$	Done
$grob24(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$	Done
$zeros(grob21(\lambda), \lambda)$	$\{-9 \cdot \sqrt{2} - 13, 9 \cdot \sqrt{2} + 13, 42.0293624379\}$
$\lambda3_12dig := 42.0293624379$	This is $\lambda3$ using 12 digits
$grob22(y, \lambda3_12dig)$	$-0.000000030000 \cdot y$
$zeros(-3 \cdot E-8 \cdot y, y)$	Solving $grob22(y, \lambda3) = 0$ yields $y = 0$. $\{0\}$
$zeros(grob21(\lambda), \lambda)[3]$	42.0293624379
$\lambda3_14dig := 42.029362437902$	This is $\lambda3$ using 14 digits
$grob22(y, \lambda3_14dig)$	$0.000000010000 \cdot y + 0.000000010000$
$zeros(1 \cdot E-8 \cdot y + 1 \cdot E-8, y)$	Solving $grob22(y, \lambda3) = 0$ yields $y = -1$. $\{-1.000000000000\}$

And because of the 12 digit limitation, the value 0 was retained. So let's set $y3 = 0$ and

$\lambda3 = 42.0293624379$. We substitute into $grob24$ and solve for x :

$grob21(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$	Done
$grob22(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$	Done
$grob23(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 1280903$	Done
$grob24(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$	Done
$y3 := 0$	0
$\lambda3 := 42.0293624379$	42.0293624379
$solve(grob24(x, y3, \lambda3) = 0, x)$	$x = 4.83346915969$

So, this is a *possible* explanation for the bug when Lagrange multipliers technique was applied to the original polynomial system:

$sys1 := \{4 \cdot x^3 - 2 \cdot \lambda \cdot x - 4 \cdot y + 2 \cdot \lambda - 4 \cdot x + 4 \cdot y^3 - 2 \cdot \lambda \cdot y + 2 \cdot \lambda \cdot x^2 - 2 \cdot x + y^2 - 2 \cdot y - 7\}$	$\{4 \cdot x^3 - 2 \cdot \lambda \cdot x - 4 \cdot y + 2 \cdot \lambda - 4 \cdot x + 4 \cdot y^3 - 2 \cdot \lambda \cdot y + 2 \cdot \lambda \cdot x^2 - 2 \cdot x + y^2 - 2 \cdot y - 7\}$
$zeros(sys1, \{x, y, \lambda\})$	$\begin{bmatrix} 4.83347 & 0 & 42.0294 \\ \frac{3 \cdot \sqrt{2} + 2}{2} & \frac{3 \cdot \sqrt{2} + 2}{2} & 9 \cdot \sqrt{2} + 13 \\ -\frac{(3 \cdot \sqrt{2} - 2)}{2} & -\frac{(3 \cdot \sqrt{2} - 2)}{2} & -(9 \cdot \sqrt{2} - 13) \end{bmatrix}$
approx	$\begin{bmatrix} 4.8334691595878 & 0 & 42.029362437902 \\ \frac{3 \cdot \sqrt{2} + 2}{2} & \frac{3 \cdot \sqrt{2} + 2}{2} & 9 \cdot \sqrt{2} + 13 \\ -\frac{(3 \cdot \sqrt{2} - 2)}{2} & -\frac{(3 \cdot \sqrt{2} - 2)}{2} & -(9 \cdot \sqrt{2} - 13) \end{bmatrix}$

We will never know if Nspire CAS has used the Gröbner basis “grob2”. But if it has been the case, Nspire CAS “has forgotten” to use the element $grob23$...

This is where the two missing solutions are coming from and, to be consequent, Nspire CAS should also have found $grob23(y3, \lambda3) = 0$:

$grob21(\lambda) := \lambda^5 - 82 \cdot \lambda^4 + 2099 \cdot \lambda^3 - 18980 \cdot \lambda^2 + 57804 \cdot \lambda - 14364$	Done
$grob22(y, \lambda) := y \cdot (6 \cdot \lambda^3 - 336 \cdot \lambda^2 + 3816 \cdot \lambda - 12312) - \lambda^4 + 63 \cdot \lambda^3 - 1028 \cdot \lambda^2 + 6504 \cdot \lambda - 14364$	Done
$grob23(y, \lambda) := 2411388 \cdot y^2 + y \cdot (401898 \cdot \lambda^2 - 20094900 \cdot \lambda + 122980788) - 6077 \cdot \lambda^4 + 388353 \cdot \lambda^3 - 5751472 \cdot \lambda^2 + 4039974 \cdot \lambda + 1280903$	Done
$grob24(x, y, \lambda) := 401898 \cdot x + 401898 \cdot y - 519 \cdot \lambda^4 + 39130 \cdot \lambda^3 - 826797 \cdot \lambda^2 + 4117814 \cdot \lambda - 158634$	Done
$y3 := 0$	0
$\lambda3 := 42.0293624379$	42.0293624379
$solve(grob24(x, y3, \lambda3) = 0, x)$	$x = 4.83346915969$
$grob23(y3, \lambda3)$	8072707.05469
$solve(grob23(y, \lambda3) = 0, y)$	$y = 0.837854239197$ or $y = 3.99561492057$

Conclusion

Without a continuous use of computer algebra in our teaching, these problems would not have been related. Moreover, our examples showed that MORE mathematics instead of LESS mathematics can be taught when computer algebra is available on the desk for each student.

We do hope that a (built-in) “Gröbner basis function” will soon be available for the Nspire CAS users. In the case of a polynomial system having a finite number of solutions, example 2 showed the importance of this kind of function because we needed to deal first with a univariate polynomial.

So, not only can we face heavy computational problems, but we can also explore different areas.

But (as a suggestion for TI) having access to more digits of precision (on the software version) would be useful.

Thank You!

Don't miss



1 – 5 July 2014, Krems, Austria

Two Important Dates:

February 15th, 2014: Deadline for submissions

April 15th, 2014: Final day for Early Bird Registration

Dear User Forum friends:

This is a very extended User Forum due to many requests and answers. We will start with a brain teaser provided by our very enthusiastic Australian member David Halprin. I add another one which I found in the German weekly newspaper “Die Zeit” in 1986. I used it for illustrating logic operations and logic connectives. *DERIVE* was used for finding the solution. Maybe that this “Logical” could form a hint how to tackle the much more difficult problem below.

David sent also his mathematical CV – which I will leave for another DNL – and an updated version of his paper which was a reaction on Fred Tydeman’s problem (DNL#87). In DNL#88 David promised a paper on this subject. The final version *Recursive Series of Numbers* has just arrived.

WHOSE FISH?

This brainteaser, reportedly written by Albert Einstein is difficult and Einstein said that 98% of the people in the world could not figure it out. Which percentage are you in? There are five houses in a row in different colors. In each house lives a person of a different nationality. The five owners drink a different drink, smoke a different brand and keep a different pet, one of which is a Walleye Pike.

Here are the hints. The question is, who owns the fish?

01. The Brit lives in the red house.
02. The Swede keeps dogs as pets.
03. The Dane drinks tea.
04. The green house is on the left of the white house.
05. The green house owner drinks coffee.
06. The person who smokes Pall Malls keeps birds.
07. The owner of the yellow house smokes Dunhills.
08. The man living in the house right in the center drinks milk.
09. The man who smokes Blends lives next to the one who keeps cats.
10. The Norwegian lives in the first house.
11. The man who keeps horses lives next to the one who smokes Dunhills
12. The owner who smokes Bluemasters drinks beer.
13. The German smokes Princes.
14. The Norwegian lives next to the blue house.
15. The man who smokes Blends has a neighbor who drinks water.

There are no tricks, pure logic will get you the correct answer. And yes, there is enough information to arrive at the one and only correct answer. If you get the correct answer, congratulations, you are one of the exclusive group of 121,348,731 people in the world who can.

Hier ist die originale Version:

Die Fahne von Nüfferland

Im Parlament des neu gegründeten Staates Nüfferland geht es hoch her. Man diskutiert über die zukünftige Landesfahne. Auf zweierlei hatte man sich schon geeinigt: auf die Farben und auf die Gestaltung der Fahne. Sie soll vier Farben haben, nämlich **schwarz** (wie die Seelen der Feinde Nüfferlands), **gelb** (wie der Sand, den man statt des erhofften Goldes hervorgebuddelt hat), **blau** (wie die Augen des Staatspräsidenten NÜFFER) und **grün** (wie die Entengrütze auf dem Teich des Regierungsgeländes). Und es soll eine Fahne mit vier untereinander liegenden Streifen sein, davon jeder in einer anderen Farbe. Die Debatte endet mit fünf Forderungen der im Parlament vertretenen Parteien:

(1) Die Radikalkonservativen fordern:

Wenn der dritte Streifen (von oben) nicht grün ist, und der vierte nicht blau, dann muss der erste entweder schwarz oder gelb sein.

(2) Die Liberaldemagogen verlangen:

Wenn der vierte Streifen weder blau noch grün ist, dann muss der zweite Streifen schwarz oder blau sein.

(3) Die Sozialkapitalisten stellen als Bedingung:

Wenn der dritte Streifen nicht grün ist, und der erste nicht blau, dann muss der gelbe Streifen entweder der zweite oder der vierte sein.

(4) Die Nationalkommunarden erheben zum Gebot:

Wenn der vierte Streifen weder schwarz noch grün ist, dann muss der erste schwarz oder gelb sein.

(5) Die NFD (Nüfferländer für Demokratur) schliesslich verlangt:

Wenn der zweite Streifen nicht schwarz und der dritte nicht gelb ist, dann muss der erste Streifen blau oder grün sein.

Staatspräsident NÜFFER findet die einzige Lösung und dekretiert sie zur Landesfahne. Wie sieht sie aus?

Aus "Die Zeit" (im Jahre 1986)

Im ersten Teil der DERIVE-Datei nueff.dfw finden Sie die deutschsprachige Durchführung aus den 90iger-Jahren. Im englischsprachigen Teil setze ich in #31 eine Funktion aus dem DNL#41 zur Gewinnung der Permutationen ein. Josef.

Ansonsten ist die Durchführung identisch mit der englischsprachigen (ab Seite 22).

Josef

This is the translated version:

The Flag of Nuefferland

The parliament of the young state Nuefferland is buzzing with excitement. The members of parliament are discussing the future flag of the state. There is agreement on two facts: the colors and the form. The flag shall show four horizontal stripes with different colors. The colors are **black** (like the souls of Nuefferland's enemies), **yellow** (like the sand which was dug out instead of the expected gold), **blue** (like state president NUEFFER's eyes), and **green** (like the duckweed on the pond on the government site). The debate ended by fixing five demands of the five parties represented in the parliament.

President Nueffer finds the only solution and declares it to the state flag. How does it look like?

(The demands are given as comments in the following DERIVE file.)

First of all we produce all possible combinations of colors (permutations of {blue, yellow, green, black}):

```
perms(v, n, k_, n_ := 2, s_, t_) :=
  Prog
    s_ := [[v↓1]]
    n := DIM(v)
    Loop
      If n_ > n exit
      k_ := n_
      #31: t_ := []
      Loop
        If k_ = 0 exit
        t_ := APPEND(t_, VECTOR(INSERT(v↓n_, v_, k_), v_, s_))
        k_ := k_ - 1
        s_ := t_
        n_ := n_ + 1
      s_
    s_

#32: flags := perms([blue, yellow, green, black])
```

We consider the demands of the parties one after the other in order to make the process clear.

Finally we will show that we can collect all demands into one logic connective.

The future flag is vector v containing four components which have to fulfill the demands which are expressed as logic expressions

(1) The demand of the Radical-Conservative Party:

If the third stripe (from above) is not green and the fourth not blue, then the first one must be black or yellow.

#33: $\text{dem1} := \neg v_3 = \text{green} \wedge \neg v_4 = \text{blue} \rightarrow v_1 = \text{black} \vee v_1 = \text{yellow}$

#34: $(\text{flags1} := \text{SELECT}(\text{dem1}, v, \text{flags})) = \text{flags1} :=$

blue	yellow	green	black
yellow	blue	green	black
yellow	green	blue	black
yellow	blue	black	green
yellow	green	black	blue
green	yellow	black	blue
yellow	black	blue	green
blue	black	green	yellow
yellow	black	green	blue
green	black	yellow	blue
black	blue	yellow	green
black	yellow	blue	green
black	blue	green	yellow
black	yellow	green	blue
black	green	blue	yellow
black	green	yellow	blue

#35: $\text{DIM}(\text{flags1}) = 16$

(2) The Liberaldemagogues demand:

If the fourth stripe is neither blue nor green, then the second one must be black or blue.

#36: $\text{dem2} := \neg (v_4 = \text{blue} \vee v_4 = \text{green}) \rightarrow v_2 = \text{black} \vee v_2 = \text{blue}$

#37: $(\text{flags2} := \text{SELECT}(\text{dem2}, v, \text{flags1})) = \text{flags2} :=$

yellow	blue	green	black
yellow	blue	black	green
yellow	green	black	blue
green	yellow	black	blue
yellow	black	blue	green
blue	black	green	yellow
yellow	black	green	blue
green	black	yellow	blue
black	blue	yellow	green
black	yellow	blue	green
black	blue	green	yellow
black	yellow	green	blue
black	green	yellow	blue

#38: $\text{DIM}(\text{flags2}) = 13$

(3) The demand of the Social-Capitalists is:

If the third stripe is not green and the first one is not blue, the yellow stripe must be on position two or four.

```
#39: dem3 :=  $\neg v_3 = \text{green} \wedge \neg v_1 = \text{blue} \rightarrow v_2 = \text{yellow} \vee v_4 = \text{yellow}$ 
```

```
#40: flags3 := SELECT(dem3, v, flags2)
```

```
#41: flags3 :=  $\begin{bmatrix} \text{yellow} & \text{blue} & \text{green} & \text{black} \\ \text{green} & \text{yellow} & \text{black} & \text{blue} \\ \text{blue} & \text{black} & \text{green} & \text{yellow} \\ \text{yellow} & \text{black} & \text{green} & \text{blue} \\ \text{black} & \text{yellow} & \text{blue} & \text{green} \\ \text{black} & \text{blue} & \text{green} & \text{yellow} \\ \text{black} & \text{yellow} & \text{green} & \text{blue} \end{bmatrix}$ 
```

```
#42: DIM(flags3) = 7
```

(4) What do the National-Communards want?

If the last stripe is neither black nor green, then the first one must be black or yellow.

```
#43: dem4 :=  $\neg (v_4 = \text{black} \vee v_4 = \text{green}) \rightarrow v_1 = \text{black} \vee v_1 = \text{yellow}$ 
```

```
#44: flags4 := SELECT(dem4, v, flags3)
```

```
#45: flags4 :=  $\begin{bmatrix} \text{yellow} & \text{blue} & \text{green} & \text{black} \\ \text{yellow} & \text{black} & \text{green} & \text{blue} \\ \text{black} & \text{yellow} & \text{blue} & \text{green} \\ \text{black} & \text{blue} & \text{green} & \text{yellow} \\ \text{black} & \text{yellow} & \text{green} & \text{blue} \end{bmatrix}$ 
```

(5) Finally the NFD (Nuefferlaenders for Democraship) demand:

If the second strip is not black and the third is not yellow, then the first one must be blue or green.

```
#46: dem5 :=  $\neg v_2 = \text{black} \wedge \neg v_3 = \text{yellow} \rightarrow v_1 = \text{blue} \vee v_1 = \text{green}$ 
```

```
#47: SELECT(dem5, v, flags4) = [[yellow, black, green, blue]]
```

Let's do it in one command:

```
#48: SELECT(dem1  $\wedge$  dem2  $\wedge$  dem3  $\wedge$  dem4  $\wedge$  dem5, v, flags)
```

```
#49: SELECT(dem1  $\wedge$  dem2  $\wedge$  dem3  $\wedge$  dem4  $\wedge$  dem5, v, flags) = [[yellow, black, green, blue]]
```



When I did this the first time #49 needed 78 seconds calculation time. Now *DERIVE* shows 0.0 sec calculation time!!

As I wrote in the Letter of the Editor, I reminded David on his intended paper on *Recursive Series*. He sent the paper together with announcing another exciting idea.

Josef,

I have re-edited the paper, cutting out some flawed material and correcting a few errors. Hopefully it is now O.K. The Derive GAS.mth allows one to do some simple calculations of the n th. term of the Fibonacci & Lucas types. Thanks again for reminding me.

I have had the bad habit of working on a topic and finding that the paper just grows and grows, so that I don't know where to complete it, so I set it aside and go onto another topic.

So my new year resolution is to complete a few other papers.

z.b. There is one incomplete paper, that is analogous to good science fiction, in that it is a mathematical fiction. I claim to have attended a mathematical conference in the form of a Seance, conducted in the European mathepality of Mathematica, wherein the spirit forms of many notable mathematicians of past centuries attended and gave papers, based on their known interests and there is much interplay between the residents etc.

Ciao, David

Then there was Simon's problem with the resolution of a laptop screen:

Dear Mag. Josef Böhm,

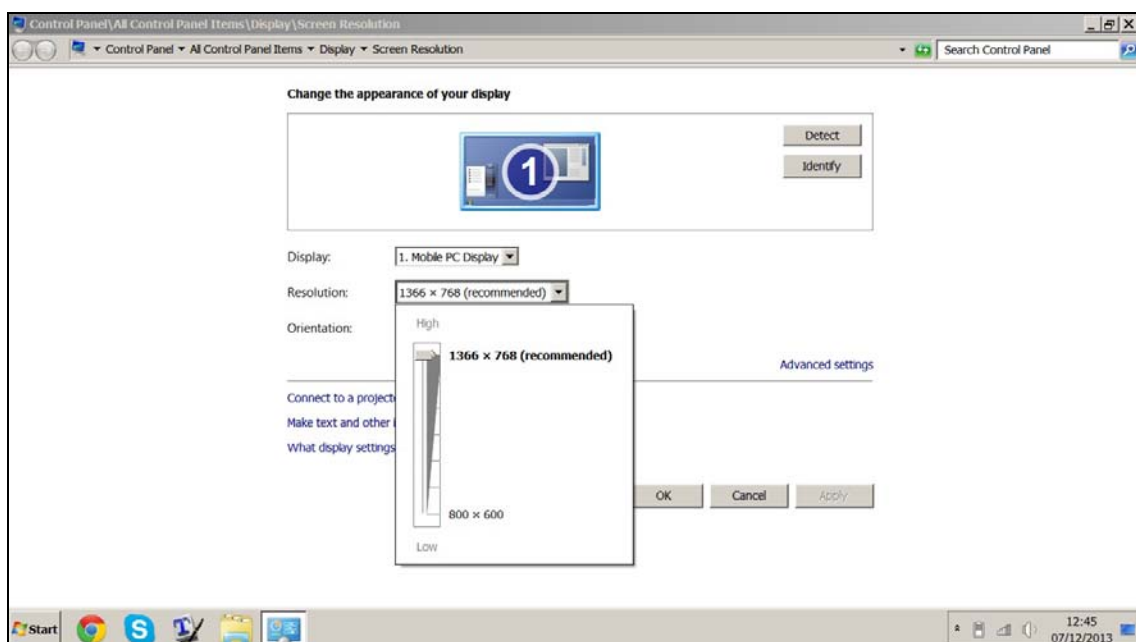
my name is Simon and I am a university student in maths.

I am experiencing the same problem as explained in D-N-L#75, page 3, by Tania Koller. That is, I get a distorted view when trying to plot in 3D.

Unfortunately, the solution proposed by her student will not work for me as 1366 x 768 is the maximal resolution for my screen (please see attachment).

Is there anything that you would suggest?

Sincerely, Simon



The best what I could do was asking our specialist for hard- and software problems, Günter Schödl. This was his (first) answer (translated by me):

Recently a professor from New Zealand wrote a mail complaining about of the same problem! DERIVE's graphic cannot cope with some widescreen resolutions. The only solution I know is to start DERIVE via a virtual system under Hyper-V (win 8 pro component) or to start Virtual Box (<https://virtualbox.org/wiki/Downloads>), where standard graphics drivers are available and presentation works in all cases!

Regards Günter

A little bit later his second advice came in:

Here I am again. I investigated the graphics problem once again. One could connect the laptop with an external fullHD screen (which is every modern monitor) then the problem should not appear. This s what my students do! Alternatively one could change the resolution (decrease it).

My mail to Simon:

Dear Simon,

I have two recommendations:

- (1) Try resolution 1024 x 768 (it works on my PC).
- (2) Connect your laptop with a common PC-screen if possible. Then it should work.

When I will work with my notebook next time – maybe today night – I will try several resolutions.

In the worst case you should change – if possible – your graphics card.

Hope this helps, let me know.
Josef

Thank you Josef.

Recommendation (1) worked.

Resolution 1024 x 768 is really low for me, but what I'll do is simply switch to that resolution to use Derive and then switch back to my native resolution when I'm done.

Thank you very much,
Simon

\iint Integrals – Integrals – Integrals \iint

During the last months some interesting problems with integrals arrived in my mailbox. Günter Schödl came across strange results working on standard problems in class room. His students modelled a wine glass with given attributes (I skip the text). The contour is described as a piecewise defined function consisting of two straight lines and a parabola.

Mail from Günter Schödl

Dear Josef!

I am working with my class on the appreciated wine glasses. The attached file shows a strange result:

When calculating the area of the cross section the integral gives different results for simplification and approximation.

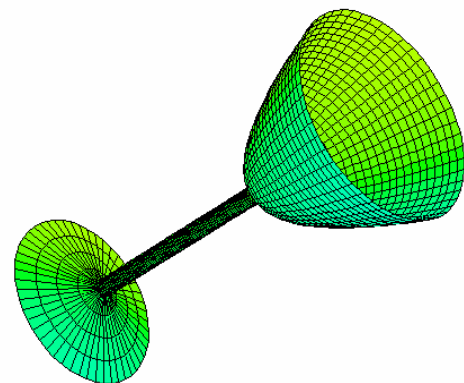
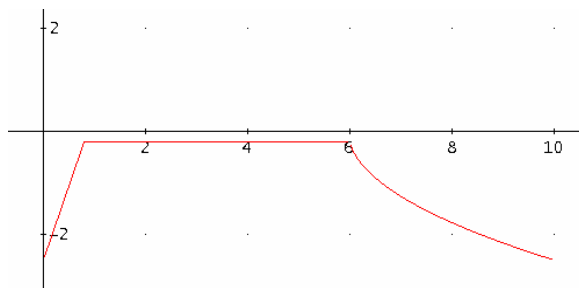
Do you have any explanation for this?

Best regards Günter

$$\left[f_1(x) := \frac{23 \cdot x}{8} - \frac{5}{2}, f_2(x) := -0.2, f_3(x) := -\frac{\sqrt{(621 \cdot x - 3710)}}{20} \right]$$

```
f(x) :=
  If 0 < x < 0.8
    f1(x)
  If 0.8 < x < 6
    f2(x)
  If 6 < x < 10
    f3(x)
  ?
```

$$[s, f(s) \cdot \cos(t), f(s) \cdot \sin(t)]$$



$$\#4: \pi \cdot \int_6^{10} f(x)^2 dx = \pi \cdot \int_6^{10} \text{IF}(6 < x < 10, f_3(x), ?)^2 dx$$

$$\#5: \pi \cdot \int_6^{10} f(x)^2 dx = 39.52123558$$

The volume of the cup
(only approximated!)

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#6: First simplifying the integral:

$$\#7: \int_0^{10} f(x) dx = - \frac{12500}{1863}$$

$$\#8: \int_0^{10} f(x) dx = -6.709608158$$

#9: And then approximating it:

$$\#10: \text{APPROX} \left(\int_0^{10} f(x) dx \right) = -8.826172856$$

#11: Or using the \approx :

$$\#12: \int_0^{10} f(x) dx$$

$$\#13: -8.826172856$$

#14: We receive different areas!! Why that?

#15: Now we calculate the area piecewise:

$$\#16: \int_0^{0.8} f_1(x) dx + \int_{0.8}^6 f_2(x) dx + \int_6^{10} f_3(x) dx = - \frac{17873}{2025}$$

$$\#17: \int_0^{0.8} f_1(x) dx + \int_{0.8}^6 f_2(x) dx + \int_6^{10} f_3(x) dx = -8.826172839$$

#18: **We get the right value in both ways, very strange!**

I must admit that I didn't know any explanation? Do you?

All what I did was defining the contour using *DERIVE*'s CHI-function – and this worked as you can see below.

Günter wrote back that he had tried with CHI, too and he had received the correct result. But the question about the strange result from above remained still open.

$$h(x) := \chi(0, x, 0.8) \cdot f_1(x) + \chi(0.8, x, 6) \cdot f_2(x) + \chi(6, x, 10) \cdot f_3(x)$$

$$\int_0^{10} h(x) dx = - \frac{17873}{2025}$$

$$\int_0^{10} h(x) dx = -8.826172839$$

$$\text{APPROX} \left(\int_0^{10} h(x) dx \right) = -8.826172856$$

$$\pi \cdot \int_6^{10} h(x)^2 dx = \frac{629 \cdot \pi}{50}$$

The volume of the cup
(now simplified!)

So I asked Michel Beaudin – and some hours later I (we) got a satisfying answer:

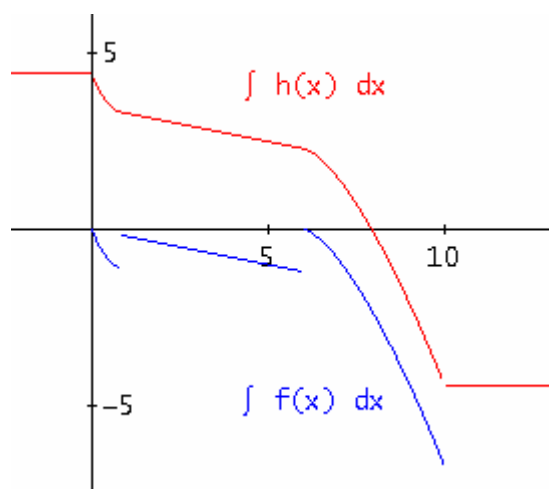
[Michel's answer](#)

Dear Josef, here is my explanation:

When Derive computes (in exact mode using "simplify" the integral of $g(x)$ between 0 and 10, Derive is using an antiderivative for $f(x)$ but the antiderivative Derive is using is discontinuous!!! But the antiderivative for $h(x)$ is continuous everywhere. See the file (in the 2D plot window, I checked "simplify before plotting" and you can see that one antiderivative is discontinuous.

BTW, Nspire CAS gives the correct result (exact or approx), using the « template » to define the piecewise function.

Michel

[My mail to Günter and Michel](#)

Many thanks to Michel for clarifying this. In my opinion this is a great example for the importance of the integration constant.

[Günter's reaction](#)

Super! Thanks for solving the mystery.

Today I discussed the problem of the antiderivative. It is very interesting and useful from didactical point of view.

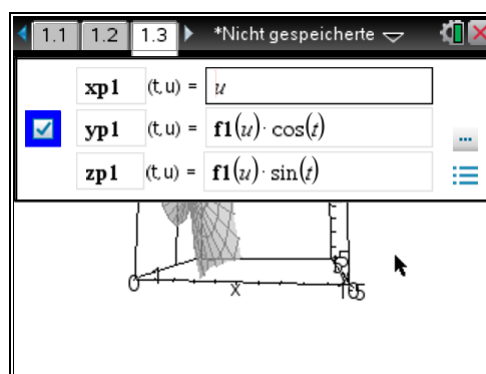
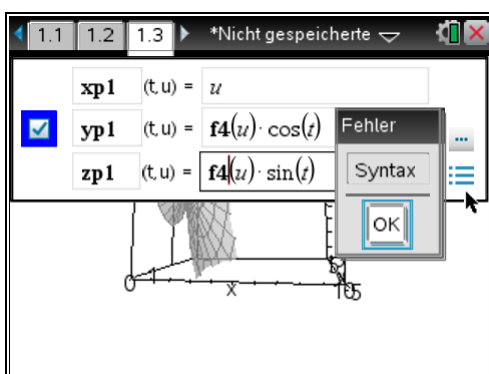
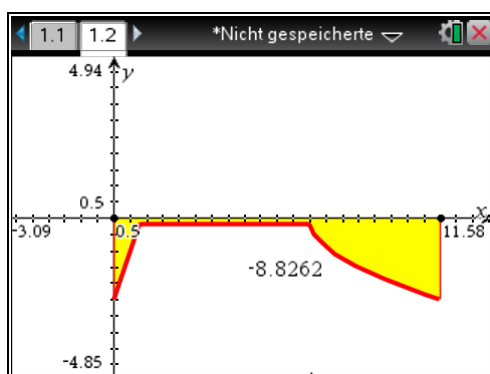
I add how this problem is worked through with TI-NspireCAS.

TI-NspireCAS screen showing the definition of piecewise functions:

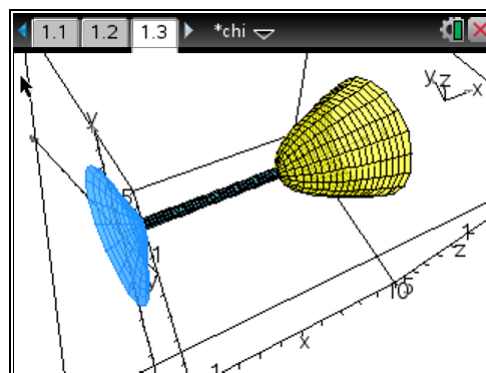
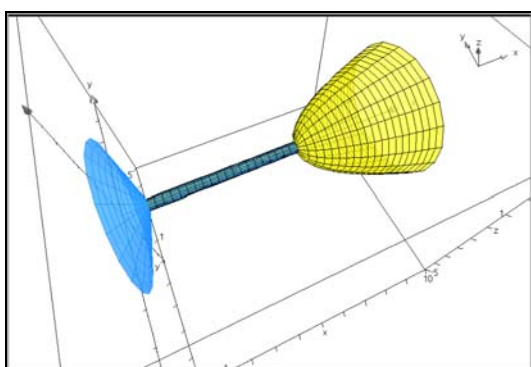
- $f1(x) = \frac{23 \cdot x}{8} - \frac{5}{2}$ Fertig
- $f2(x) = -0.2$ Fertig
- $f3(x) = \frac{-\sqrt{621 \cdot x - 3710}}{20}$ Fertig
- $f4(x) = \begin{cases} f1(x), & 0 \leq x < 0.8 \\ f2(x), & 0.8 \leq x < 6 \\ f3(x), & 6 \leq x \leq 10 \end{cases}$ Fertig

TI-NspireCAS screen showing the integration of $f4(x)$ from 0 to 10:

- Integration template: $\int_0^{10} f4(x) dx$
- Result: $\frac{-17873}{2025}$
- Approximate result: -8.82617



It is not possible to use the piecewise defined function to define the parameter form of the solid of rotation. One has to rotate f_1 , f_2 and f_3 in a row, then it works and one will get nice graphs on the PC-screen and on the handheld as well.



I have tried with other piecewise defined functions, too. Then it has worked. It seems to be that the fact that the square root give complex values in the interval $[0,10]$ causes the problems.

And there were three more requests on integration problems, all of them presented by Francisco Marcelo Fernandez from Argentina.

They are really interesting. I compared *DERIVE*'s and the Nspire's behaviour and found some differences. *DERIVE*'s Stepwise Simplification was sometimes helpful. It is mere chance that problem (3) seems to be connected with Günter's request from above and Michel's respective answer.

(1) Dear Derivians,

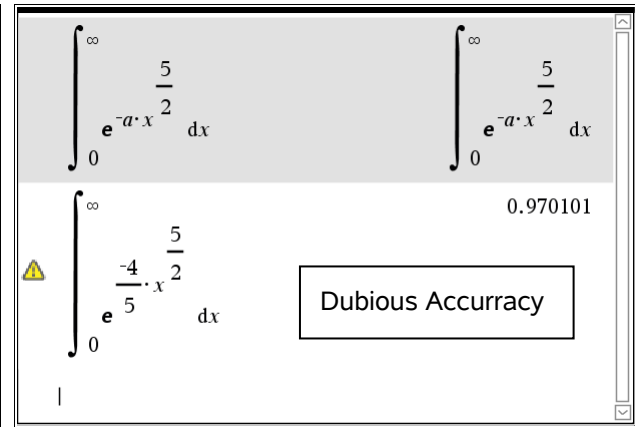
I am attaching a short dfw-file that shows a problem that I have calculating integrals that should give gamma functions. Is there any way to overcome this undesirable output?

Greetings, Francisco

#1: $a \in \text{Real } (0, \infty)$

$$\#2: \int_0^{\infty} \text{EXP}(-a \cdot x^{5/2}) dx = \frac{\left(\frac{2}{5}\right)!}{a^{2/5}}$$

$$\#3: \int_0^{\infty} \text{EXP}\left(-\frac{4}{5} \cdot x^{5/2}\right) dx = \int_0^{\infty} \frac{1}{\left(\frac{x^{5/2}}{e}\right)^{4/5}} dx$$



(2) Dear Derivians,

I found this unpleasant behaviour of Derive 5 and Derive 6. Perhaps someone can explain it.

$$\text{SUBST} \left(\frac{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}, n, 1 \right) = \frac{18^{1/5} \cdot \text{COS}\left(\frac{4}{3}\right)}{3 \cdot \text{COS}\left(\frac{2}{3}\right)^{4/5}} + \frac{18^{1/5}}{3 \cdot \text{COS}\left(\frac{2}{3}\right)^{4/5}}$$

$$\text{SUBST} \left(\frac{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}, n, 2 \right)$$

$$\frac{\left(\frac{(7500 \cdot \sqrt{5} - 12500)^{1/7}}{10} + \frac{(60 - 20 \cdot \sqrt{5})^{1/7}}{2} \right) \cdot \text{COS}\left(\frac{4}{5}\right)}{\text{COS}\left(\frac{2}{5}\right)^{4/7}} + \frac{\left(\frac{(7500 \cdot \sqrt{5} - 12500)^{1/7}}{10} + \frac{(60 - 20 \cdot \sqrt{5})^{1/7}}{2} \right)}{\text{COS}\left(\frac{2}{5}\right)^{4/7}}$$

1.899638696

My tries:

Substitution for $n = 1, 2$ works properly in Exact Mode. Increasing n leads to strange results:

$$\text{SUBST} \left(\frac{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}, n, 3 \right) = (1, 0)$$

$$\text{APPROX} \left(\text{SUBST} \left(\frac{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}{\left(\frac{\text{SIN}\left(\frac{\pi}{2 \cdot n + 1}\right)}{\text{COS}\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1)/(2 \cdot n + 3)}}, n, 3 \right) \right) = 3.436592793$$

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$$\text{SUBST} \left(\frac{\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\cos\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}}{\cos\left(\frac{2}{2 \cdot n + 1}\right)}, n, 4 \right) = (1, 0)$$

$$\text{APPROX} \left(\text{SUBST} \left(\frac{\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\cos\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}}{\cos\left(\frac{2}{2 \cdot n + 1}\right)}, n, 4 \right) \right) = 5.556064454$$

It is interesting that applying the VECTOR-command other simplification takes place (n = 1, 2). Compare with the outcomes above!

$$\text{VECTOR} \left(\frac{\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\cos\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}}{\cos\left(\frac{2}{2 \cdot n + 1}\right)}, n, 2 \right) = \left[\frac{2 \cdot 18^{1/5} \cdot \cos\left(\frac{2}{3}\right)^{6/5}}{3}, \frac{2 \cdot (27500 \cdot \sqrt{5} + 62500)^{1/7} \cdot \cos\left(\frac{2}{5}\right)^{10/7}}{5} \right]$$

$$\text{VECTOR} \left(\frac{\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\cos\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}}{\cos\left(\frac{2}{2 \cdot n + 1}\right)}, n, 2 \right) = [0.8900113409, 1.899638696]$$

$$\text{VECTOR} \left(\frac{\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\cos\left(\frac{2}{2 \cdot n + 1}\right)} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}}{\cos\left(\frac{2}{2 \cdot n + 1}\right)}, n, 3 \right) = (1, 0)$$

Seems to be a bug!

I found out that the problem might be buried in the subexpression

$$\sin(\pi/(2 \cdot n + 1)) / \cos(2/(2 \cdot n + 1))$$

Applying stepwise simplification for n > 2 leads to 0 ^ n --> +- inf.

But if you rewrite the expression substituting the denominator with

$$\sqrt{1 - \sin(2/(2 \cdot n + 1))^2}$$

then everything is ok - having set the Mode Trigonometry > Expand > Sines.

Trying now stepwise simplification you will recognize just another simplification procedure.

Don't know why!

Maybe that Albert Rich is checking the mail and he could be the only one to unveil this secret!??

$$\#19: \frac{\sin\left(\frac{\pi}{7}\right)}{\cos\left(\frac{2}{7}\right)}$$

If n < 0 is odd,

$$0^n \rightarrow \pm\infty$$

$$\#20: (1, 0)$$

$$\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\sqrt{1 - \sin\left(\frac{2}{2 \cdot n + 1}\right)^2}} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}$$

[Trigonometry := Expand, Trigpower := Sines]

$$\frac{2^{2/9} \cdot \left(\cos\left(\frac{4}{7}\right) + 1\right)^{7/9}}{2 \cdot \sin\left(\frac{\pi}{7}\right)^{14/9}}$$

3.436592793

$$\text{SUBST} \left(\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\sqrt{1 - \sin\left(\frac{2}{2 \cdot n + 1}\right)^2}} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}, n, 3 \right) = \frac{2^{35/36} \cdot \sin\left(\frac{\pi}{7}\right)^{4/9} \cdot \left(\cos\left(\frac{4}{7}\right) + 1\right)^{7/9}}{2^{3/4} - \sqrt{\sqrt{2} - \sqrt{1 - \cos\left(\frac{\pi}{7}\right)}}}$$

$$\text{SUBST} \left(\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\sqrt{1 - \sin\left(\frac{2}{2 \cdot n + 1}\right)^2}} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}, n, 3 \right) = 3.436592793$$

$$\text{SUBST} \left(\left(\frac{\sin\left(\frac{\pi}{2 \cdot n + 1}\right)}{\sqrt{1 - \sin\left(\frac{2}{2 \cdot n + 1}\right)^2}} \right)^{-2 \cdot (2 \cdot n + 1) / (2 \cdot n + 3)}, n, 4 \right) = \frac{2^{2/11} \cdot \sin\left(\frac{\pi}{9}\right)^{4/11} \cdot \left(\cos\left(\frac{4}{9}\right) + 1\right)^{9/11}}{1 - \cos\left(\frac{2 \cdot \pi}{9}\right)}$$

Another strange thing happened. I saved the file and when reloading it I came across a "parsing error". You can see the "result" (1,0). But (1,0) is not a valid *DERIVE* expression. So *DERIVE* does not even recognize its selfmade output ... - this is more than strange.

I tested with both Voyage 200 and TI-NspireCAS:

The screenshot shows the TI-NspireCAS interface. The top part displays a complex expression involving trigonometric functions and exponents. Below it, the same expression is shown with a yellow triangle icon, indicating a warning or error. The bottom part shows the expression simplified for n=3, resulting in a numerical value of 14/9.

First surprise: How can we explain Nspire's simplification (rewriting) of the expression?

Second surprise: How can we prove the identity of the results for n = 3 or n = 4?

I checked the identity (for n=3) by approximation and tried to simplify the difference of the results for n=4 to zero, but failed in Exact Mode. Approximation gives a “tiny” result close to zero.

What's about good old Voyage 200: It performs – not unexpected – the same simplification and does not give any problem in simplifying the substitutions.

(Where does the $4/(2n+3)$ come from?)

And there was a third integral problem presented by Francisco:

(3) Dear Derivians,

I am attaching a short dfw file that shows a problem that I have calculating some kind of integrals. They appeared as part of a larger calculation. The same problem appears when using Derive 6. I solve the problem by substituting the result into the expression where it appears but this process make the calculation slower. Any way to do it automatically?

Greetings, Francisco

Derive has no problem calculating this integral

$$\#1: \int_0^{\infty} \text{EXP}(-|x-1|) dx = e^{-1} \cdot (2 \cdot e - 1)$$

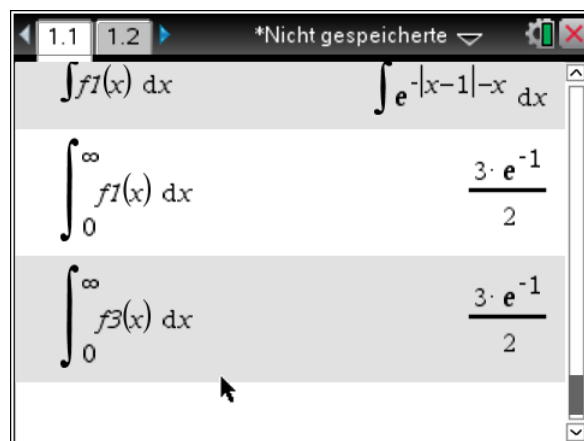
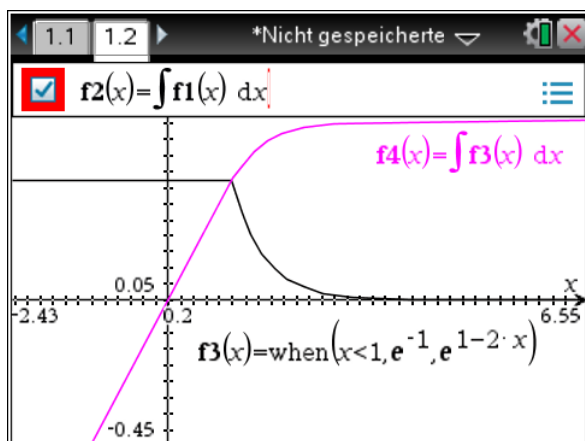
but somehow it is unable to do

$$\#2: \int_0^{\infty} \text{EXP}(-|x-1| - x) dx = \int_0^{\infty} e^{-|x-1| - x} dx$$

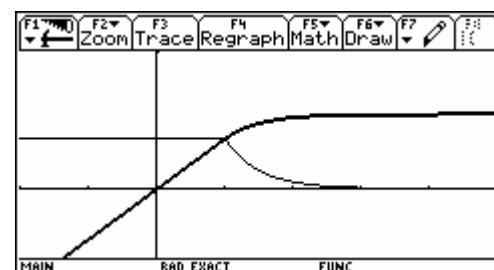
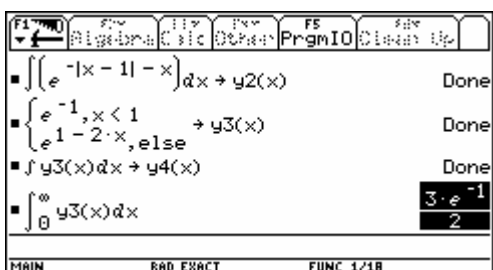
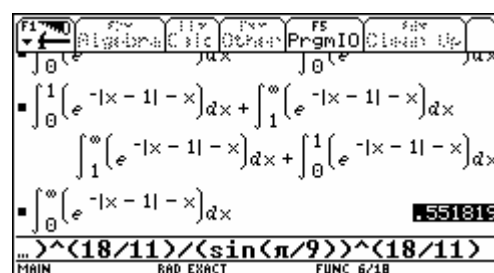
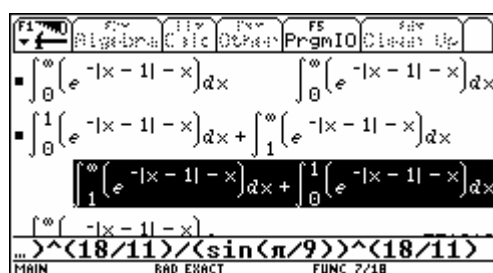
It can do it with some help

$$\#3: \int_0^1 \text{EXP}(-|x-1| - x) dx + \int_1^{\infty} e^{-|x-1| - x} dx = \frac{3 \cdot e^{-1}}{2}$$

Let's first investigate how the TIs are handling the problem:



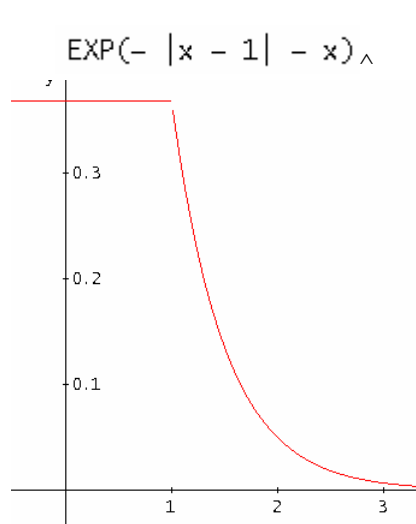
Voyage 200 is not able to integrate the function in given form in exact mode, but in “when-form” like $f_3(x)$ on the TI-Nspire.



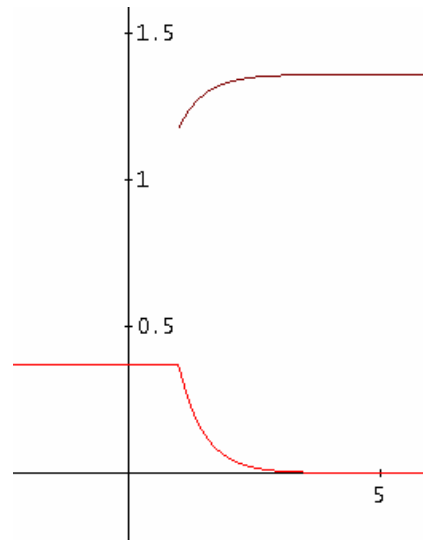
The integrals cannot be evaluated in exact mode, even not when split into two parts. See the graphs of $y_3(x)$ and $y_4(x)$ - the thick one. $y_2(x)$ cannot be plotted.

What can we do in DERIVE?

I start with plotting the function and with finding and plotting the undefined integral in order to get some insight – if possible?



$$\int \frac{\exp(-|x-1|-x) \cdot (x \cdot \text{SIGN}(x-1) + x - 1)}{\text{SIGN}(x-1) + 1} dx$$



Inspecting the integral we can see the reason for the trouble immediately. For $x < 1$ the denominator becomes zero and so the whole expression becomes undefined!

Simplify $\text{abs}(x-1)$ applying Stepwise Simplification. (Try simplifying stepwise the integral!!)

#27: $|x - 1|$

$|x| \rightarrow x \cdot \text{SIGN}(x)$

#28: $(x - 1) \cdot \text{SIGN}(x - 1)$

So, what else? No solution in sight? We define the function piecewise considering the value of $|x - 1|$ for $x < 1$ and $x \geq 1$.

#22:
$$f(x) := \begin{cases} e^{-1} & \text{If } x < 1 \\ e^{1-2x} & \text{If } x \geq 1 \end{cases}$$

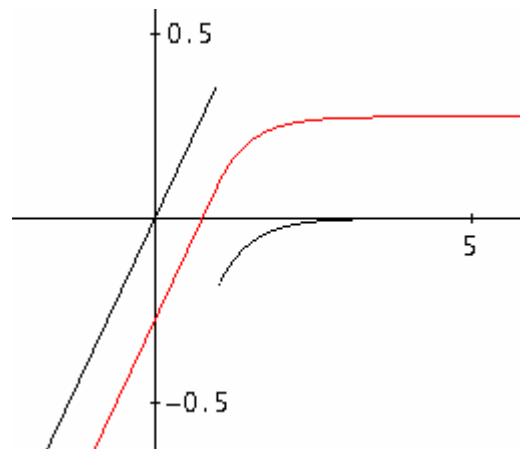
#23:
$$\int_0^{\infty} f(x) dx = 0$$

#24:
$$g(x) := \chi(-\infty, x, 1) \cdot \exp(-1) + \chi(1, x, \infty) \cdot \exp(1 - 2x)$$

#25:
$$\int_0^{\infty} g(x) dx = \frac{3 \cdot e^{-1}}{2}$$

Plots of #22 and #24 result in the correct function graph. But as you can see, only $g(x)$ leads to the correct integral.

If you have read Michel's advice (page 32) carefully then you can find the explanation by yourself. We plot again the antiderivatives of $f(x)$ – black – and $g(x)$ – red – and we face again the fact of a discontinuous function in the first case.



This is quite another question:

Dear Derivians,

I am interested in an efficient program to find a set of three numbers n_1 , n_2 , and n_3 ($n_i = 1, 2, \dots$) such that $n_1^2 + n_2^2 + n_3^2 = n$ for a given n . Mine is rather slow and inefficient. Does anyone have something like it?

Regards, Marcelo

Ignacio Larrosa Cañestro sent a short very clever program

```

S3C(n) :=
  Prog
  v := []
  If MOD(n, 8) = 7
    RETURN v
  m := FLOOR(√(n/3))
  a := 0
  Loop
    If a > m
      RETURN v
    b := a
    Loop
      If b^2 > (n - a^2)/2 exit
      c := √(n - a^2 - b^2)
      If INTEGER?(c)
        v := APPEND(v, [[a, b, c]])
      b := b + 1
    a := a + 1
  #1:
  #2: S3C(101)
  #3:
    [ 0  1 10 ]
    [ 1  6  8 ]
    [ 2  4  9 ]
    [ 4  6  7 ]

#4: S3C(1001)
#5:
    [ 1 10 30 ]
    [ 1 18 26 ]
    [ 2  6 31 ]
    [ 4 12 29 ]
    [ 4 16 27 ]
    [ 5 20 24 ]
    [ 6 17 26 ]
    [ 8 19 24 ]
    [10 15 26 ]
    [13 16 24 ]

```

Dear Josef Böhm

I am Behrooz Khavari from IRAN and have some experience with Derive. I hope the attachment can be useful. I am sorry if my English is not good.

Best regards, Behrooz

Are mathematical software products reliable!?

Behrooz Khavari , University of Iranshahr, IRAN

Abstract: Wide and fast action of mathematical software in plotting and computations, cause to unavoidable use of those in mathematical education. But such as every other scientific tool, true information about ability and infirmity of those are necessary. In this article we try to get you acquitted with such instances.

Introduction

Daily advancement and expanding applications of mathematics in other sciences persuade us to use any suitable tool for correct teaching mathematics in best possible way. Experiences and researches have shown that besides text books, correct use of educational assistant tools leads to learning mathematical concepts faster and deeper by students. Because of high accuracy and fast performance, mathematical software are the preferred option nowadays. The use of mathematical software as educational assistant tools started some decades ago. In recent times the use of those is growing up. But we should note that, using technology with hurry creates problems and mathematical software are not excepted.

Mistakes made while using mathematical software

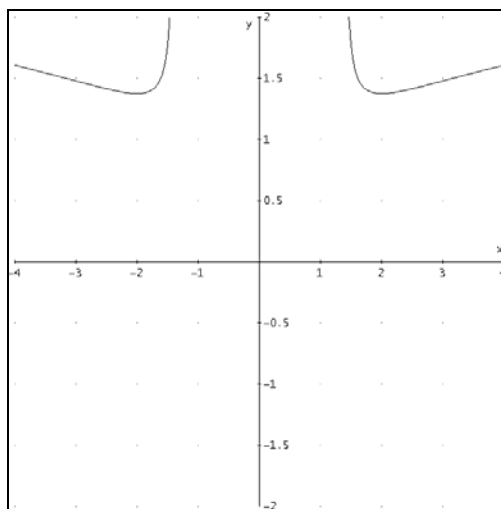
In the following, we shall be illustrating some errors and confusion encountered by individuals, while using mathematical software. We will focus solely on graph plotting because; mistakes in this content are easy to figure out. The examples are as follows:

Example 1: Let us to plot graph of the not so complicated function $f(x) = \frac{\sqrt{x^2 - 1}}{\sqrt[3]{x^2 - 2}}$. We use the mathematical software Derive, to achieve our goal. Fig 1 shows the conclusion of our first example¹.

Fig 1: Graph of $f(x) = \frac{\sqrt{x^2 - 1}}{\sqrt[3]{x^2 - 2}}$

plotted by Derive.

There is no problem, at first glance. We have a function and a mathematical software that is one of the most useful software in field of mathematics. So we use it in plotting. But, is this the graph of our function? The Answer is no because, after some calculations we arrived at this;



$$D_f = (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1] \cup [1, \sqrt{2}) \cup (\sqrt{2}, +\infty)$$

¹All graphs are plotted according to initial setting of software, otherwise mentioned.

But the domain of plotted function is only $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$. So Derive did not plot the function completely. That is, it's easy to be misled if we have no idea about the behavior or at least domain of the function from the beginning.

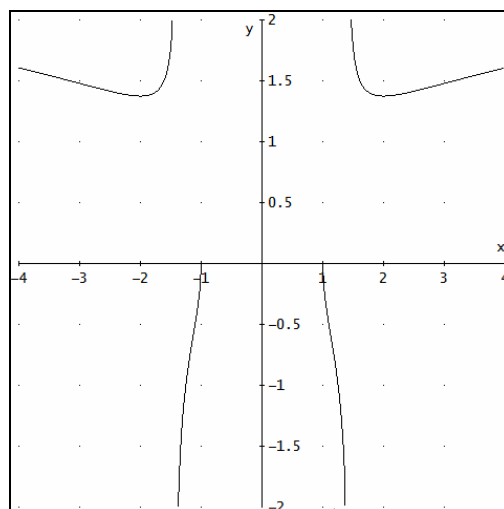
Coming across such example, especially at first, may seem very dispiriting, but do not conclude so fast. This anomalous conclusion is due to default setting of Derive and existence of $\sqrt[3]{x^2 - 2}$ at the structure of the function. This setting is such that if Derive encounter with a fractional power of an expression during a calculating process, it always consider the main branch that is the most common choice. This manner leads to obtaining complex value for $\sqrt[3]{x^2 - 2}$ on the interval $(-2, 2)$ and so $f(x)$ will be a complex-value function throughout the mentioned interval. So on this interval there is no plotting of graph.

Now to eradicate this problem it is enough that we follow *Options > Mode Settings...* in the Algebra window and then choose “real” from the *Branch* tab. After this, issuing the plot command leads to:

Fig 2: Graph of $f(x) = \frac{\sqrt{x^2 - 1}}{\sqrt[3]{x^2 - 2}}$

plotted by Derive after some proper changes in setting.

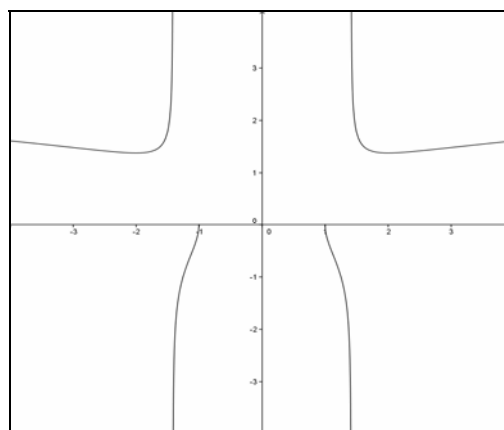
A little probing shows that the graph is plotted completely.



If we use Geogebra software, the graph will be plot at the first, completely (Fig 3).

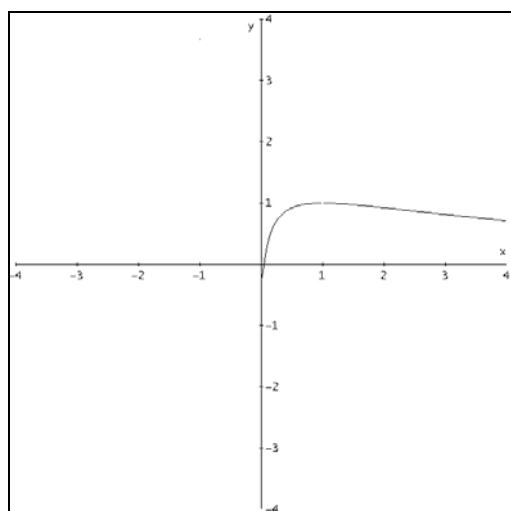
Fig 3: Graph of $f(x) = \frac{\sqrt{x^2 - 1}}{\sqrt[3]{x^2 - 2}}$

plotted by Geogebra.

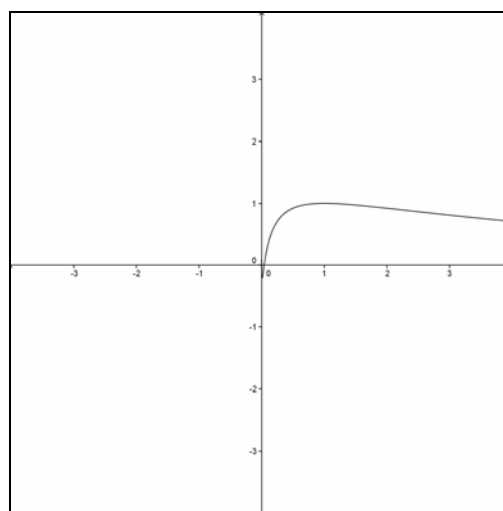


Example 2: This time we shall sketch the graph of $h(x) = \frac{\sin(\ln(x))}{\ln(x)}$. But by a little probe, before plotting, it will be clear that at least $x = 1$ does not belong to D_f . If we use Derive and Geogebra we will arrive at:

Fig 4:



Graph of $h(x) = \frac{\sin(\ln(x))}{\ln(x)}$
plotted by DERIVE



Graph of $h(x) = \frac{\sin(\ln(x))}{\ln(x)}$
plotted by GeoGebra

The graph plotted by Derive has a discontinuity at $x = 1$ (Fig 4. left), while the other is completely continuous (Fig 4. right)! So, if we do not mention to the domain of the function at first then Geogebra will confuse us!

Note: Maple, Mathematica and IT-Nspire software will sketch the graph as same with Geogebra!

Therefore, we can be easily confused with the answers we get from the mathematical software. But now, telling a promising example seems necessary. Because sometimes, an unexpected answer from a software may persuades us to expand our knowledge for detecting reality. For example, take $h(x)$ again. Imagine at first we want to find its domain without any knowledge about its graph and then plot its graph by a software for accuracy. The common way of finding domain for $h(x)$ is as follows:

With the assumption of $g(x) = \frac{\sin(x)}{x}$ and $f(x) = \ln(x)$ we have $h(x) = g(f(x))$. According to the definition of domain of the composite function:

$$D_h = D_{g \circ f} = \{x \in D_f \mid f(x) \in D_g\} = \{x > 0 \mid \ln(x) \neq 0\} = \mathbb{R}^+ - \{1\}$$

At first glance it conforms the graph which was plotted by Derive. But it is possible you may find a point with coordinate $(-1, 3.6)$, randomly (ordinate is approximate). Abscissa of this point is out of D_h . Is this a dirt on monitor or one with damaged pixels? After crosschecking we shall understood it belongs to the graph!! Can this be possible?!

Yes; it is possible. After expanding discussion to complex analysis and reference to suitable books we will detect that $h(x)$ is real for $x > 0$ and is complex for $x < 0$ unless at $x = -1$ that we have:

$$h(-1) = \frac{\sin(\ln(-1))}{\ln(-1)} = \frac{\sin(i.\pi)}{i.\pi} = \frac{i.\frac{e^{\pi} - e^{-\pi}}{2}}{i.\pi} = \frac{e^{\pi} - e^{-\pi}}{2\pi} \approx 3.676077910$$

(i is imaginary unit and e is Neperian^[*] number)

that is a real number and so -1 is belongs to D_h .

Again we mention that the Maple, Mathematics and TI-Nspire software don't show the point $(-1, 3.676077910)$.

In the next table, the results of using some online mathematical software are compared for the last example.

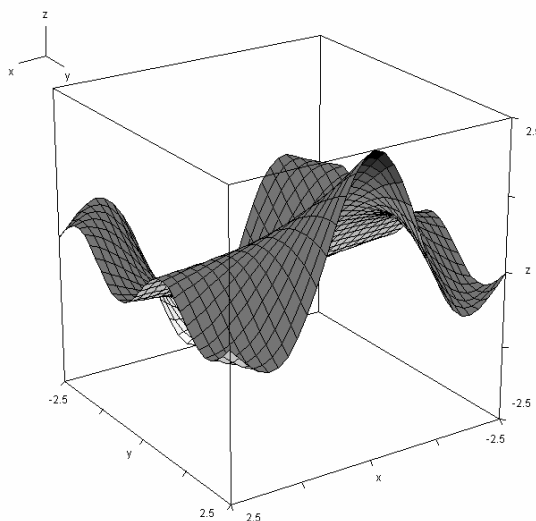
<i>link</i>	D_h	<i>explanation</i>
https://www.desmos.com/calculator	$R^+ - \{1\}$	By tracing the plot discontinuity will be shown.
http://www.fooplot.com	R^+	
http://functiongrapher.com/	R^+	
http://www.rechneronline.de/function-graphs/	$R^+ - \{1\}$	By removing the mesh, discontinuity will be shown.
http://www.onlinefunctiongrapher.com/	R^+	
http://www.gcalc.net/	$R^+ - \{1\}$	
http://my.hrw.com/math06_07/nsmedia/tools/Graph_Calculator/graphCalc.html	R^+	
http://www.mathopenref.com/graphfunction.html	R^+	
http://graphsketch.com/	R^+	
http://www.intmath.com/functions-and-graphs/graphs-using-jsxgraph.php	R^+	

Table: Comparing of results from using some online mathematical software for example 2.

[1] Did you know that the *Euler* number e is also called *Neperian* number, Josef?

Example 3: We are going to plot $f(x, y) = \frac{\sin(x, y)}{x}$. Using Derive with setting “Number of panels: (30, 30)” without changes in other setting leads to:

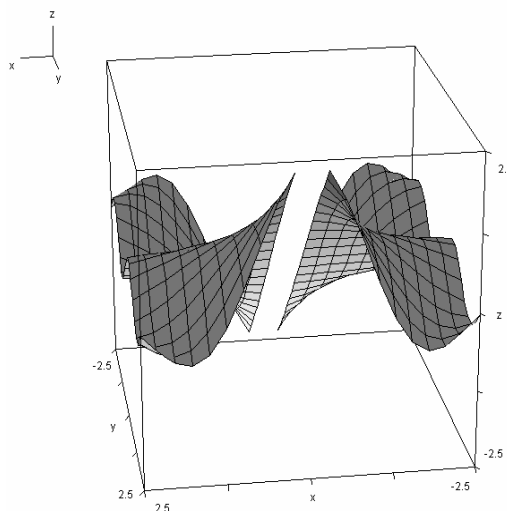
Fig 5: Graph of $f(x, y) = \frac{\sin(x, y)}{x}$
plotted by Derive when the number of
panels is (30,30).



It seems that this function is continuous on the plotted region. But it's criterion shows that the function is obviously discontinuous throughout the y axis.

Now if we change number of panels setting to (20, 20), plot the graph again and finally rotate the plotted graph to the left a little, discontinuity will appear (Fig 6).

Fig 6: Graph of $f(x, y) = \frac{\sin(x, y)}{x}$
plotted by Derive when the number of
panels is (20,20) and after a little rotation
to the left.



We've all heard and seen so many things about abilities of software such that never despond from using software because of some counterexamples. But it is better that we think about ways that leads to less mistake like that mentioned later. For this purpose, it is better to consider the following:

We've all heard and seen so many things about abilities of software such that never despond from using software because of some counterexamples. But it is better that we think about ways that leads to less mistake like that mentioned later. For this purpose, considering the following is better:

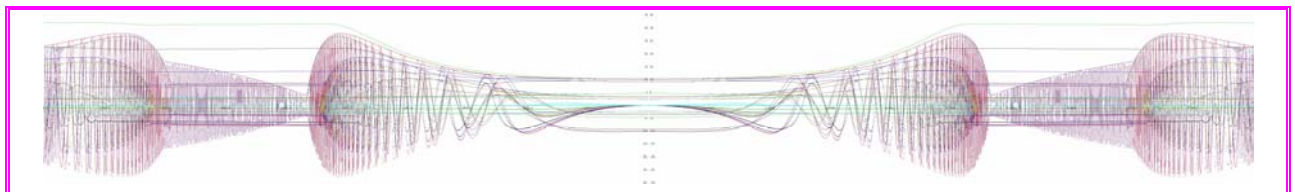
- 1) You shouldn't use software! Don't be surprised. Because sometimes using software is not logical. For example when level of your knowledge about mathematical function is up to knowing about the Sin and Ln functions it should be clear for you that $\ln(1) = 0$ and so $x = 1$ is not belongs to domain of $h(x) = \frac{\sin(\ln(x))}{\ln(x)}$ so $h(x)$ is not continuous at $x = 1$.
- 2) For more complicate problems, first try to prepare brevity of solution or probably answer in your mind; in other hand don't let to the computer that think about problems instead of you during the solution process.
- 3) It is better that you have experience with at least two software till check your problem with both, if necessary.
- 4) Sometimes changes in resolution, zoom or other setting may be helpful (same that we saw in example 1 and 3).

And finally:

***Software are good assistants that do mistake sometimes,
use them but do not rely on them.***

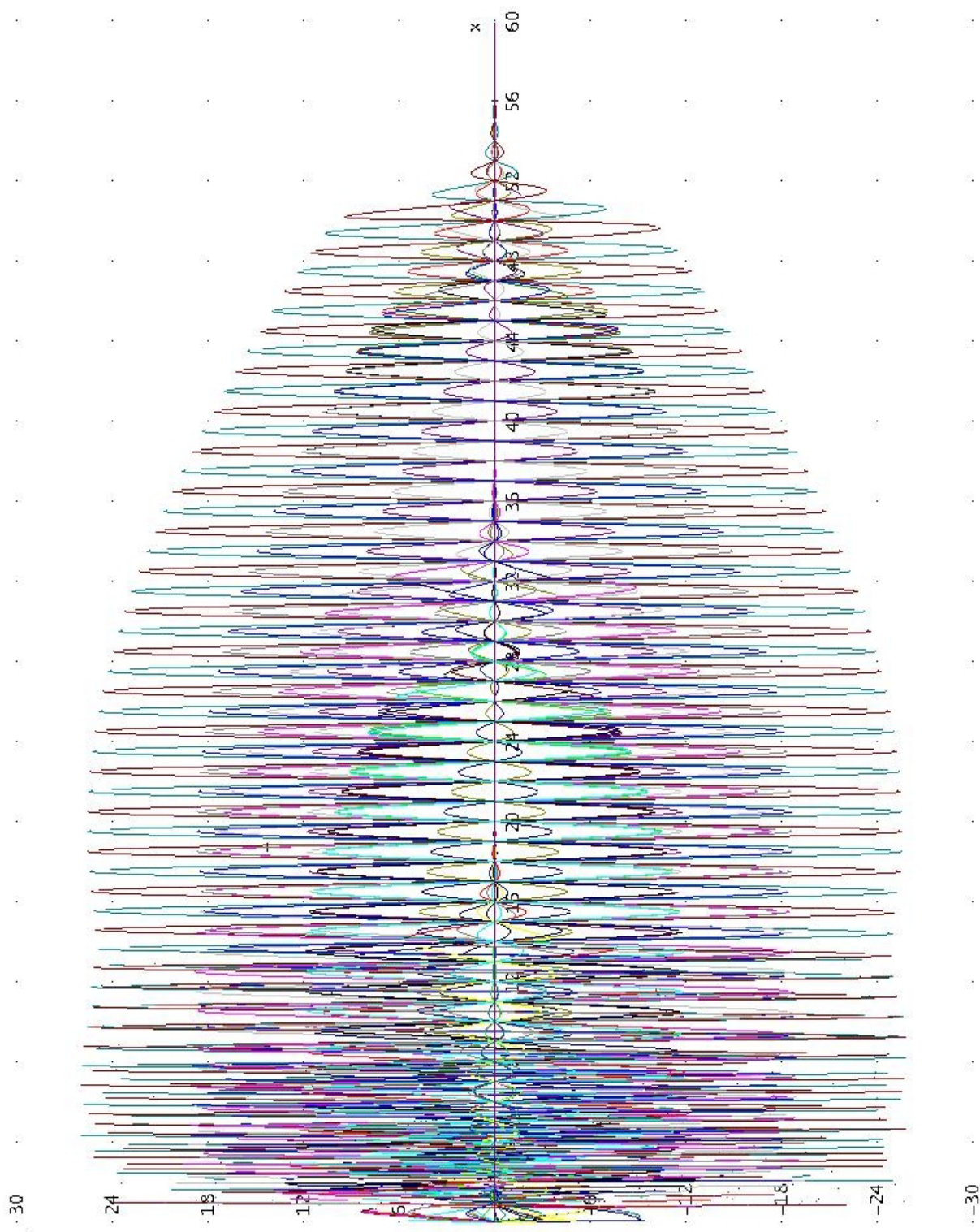
References

- 1- Derive, Ver 6.1
- 2- Geogebra, Ver 4
- 3- Maple, Ver 13
- 3- Mathematica, Ver 7
- 4- TI-Nspire, Ver 3.2



(Dietmar Oertel, December 2013)

Happy 2014



With this wonderful *DERIVE* candle (Dietmar Oertel) we are sending our best wishes for 2014 to you: Health, Peace, Happiness and Success,

Noor and Josef Böhm