Derive™ Tutorial Twelve

Linear Algebra

Eigenvalues and Eigenvectors

An eigenvector of a linear transformation is a vector that the linear transformation simply rescales and/or reverses. The scaling factor of an eigenvector is its eigenvalue. To determine the eigenvector of a linear transformation, you begin by finding its eigenvalue. You can see that the equation $A\vec{v} = \lambda\vec{v}$ indicates that $\vec{v}$ is an eigenvector of the matrix A with eigenvalue λ.

You can use Derive to find both the eigenvalues and their eigenvectors. But first you need to work with the equation $A\vec{v} = \lambda\vec{v}$. This equation is equivalent to $A\vec{v} - \lambda v = \vec{0}$ which in turn is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$. The product of a matrix and a non-zero vector is zero only if the matrix is singular. That is, only if the matrix has a zero determinant.

For example, Author each of the following:

A:=[1, 2, -1; 1, 0, 1; 4, -4, 5] (Notice the semicolons to end each row.)
A- λ *IDENTITY_MATRIX (3) (λ is found on the lower left of your screen.)
DET(A- λ * IDENTITY_MATRIX (3))
SOLVE(DET(A- λ *IDENTITY_MATRIX(3)) = 0, λ)

You can see that the eigenvalues of the matrix A are 1, 2, and 3. Another way of accomplishing the same result is to Author and Simplify the following:

charpoly(A, λ)
solve(charpoly(A, λ), λ).

You can also accomplish this result by Authoring eigenvalues(A, λ).

You can find the eigenvectors in two ways. First, you can solve the equation $A\vec{v} = \lambda\vec{v}$ for $\vec{v} = [x, y, z]$ using one of the eigenvectors, say, $\lambda = 2$. To do this, Author SOLVE(A·v = 2·v, [x, y, z]) and Simplify. The result is two equations in three unknowns. If z is a number, say, $z = -1$, then x and y can be determined. To do this using Derive, use the Simplify>Variable Substitution and in the dialog box, highlight z only, enter -1 for New Value, and click on Simplify. The values of x and y are $x = \frac{1}{2}$ and $y = -\frac{1}{4}$. Recalling that $z = -1$, the eigenvector for the eigenvalue $\lambda = 2$ is $\vec{v} = \left[\frac{1}{2}, -\frac{1}{4}, -1\right]$.

A second way to determine the eigenvectors is to use the Exact_Eigenvector command in Derive. To do this, Author Exact_eigenvector(A,2).

You can check this result by Authoring v:=[1/2,-1/4,-1] and then Authoring A*v and Simplifying. The expression $A\vec{v}$ can be simplified by hand as $A\vec{v} = \lambda\vec{v} = 2\vec{v} = 2\left[\frac{1}{2}, -\frac{1}{4}, -1\right] = \left[1, -\frac{1}{2}, -2\right]$.

Factoring Matrices

The QR and LU are two standard factorizations of a matrix. To obtain either of these factorizations, you use the Factor feature of Derive. For example, you can factor the matrix A above. First, Author A and then use the Simplify>Factor command and in the dialog box, click on the QR button in the lower right corner and then click on Factor. You can obtain the LU factorization in the same way except you click on the LU button. There are several types of LU decompositions. You should use the Online Help facility in Derive

to learn about the decomposition you want.

Change of Basis

You can use Derive's Row_Reduce command to change from one set of basis vectors to another. To do this, you construct a matrix using the two sets of basis vectors as the matrix columns. An example is given below.

Let the two sets of vectors *S* and *T* be given by

$$S:\ \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \text{and } \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \text{ and}$$

$$T:\ \vec{w}_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \text{and } \vec{w}_3 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix} \text{ (in the standard basis vectors } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\text{).}$$

You can determine the *S*-coordinates of a vector $\vec{x}$ given in the *T*-coordinates as say [1,1,2]. You do this using the columns of *S* and *T* in building a matrix. To do this,

Author Row_Reduce([3,2,4,2,-1,3; 2,1,2,1,2,1 ;-1,1,-5,3,4,-2]) and Simplify.

The last three columns of the displayed result (a 3×6 matrix) make up the transition matrix from *T* to *S*, written $P_{S \leftarrow T}$. That is to say, this transition matrix $P_{S \leftarrow T}$ when applied to a vector $\vec{x}$ represented in *T*-coordinates will give the *S*-coordinate representation of $\vec{x}$.

Author $P_{S \leftarrow T}$ * [1,1,2] (where $P_{S \leftarrow T}$ is entered as [0, 5, -1; 11/7, - 22/7, 9/7; - 2/7, - 17/7, 6/7]) to find the *S*-coordinates of the vector $\vec{x}$ given above. The representation is [3,1,-1].

The vector $\vec{x}$ can now be represented in two ways.

You can check that these two representations of $\vec{x}$ are the same by performing the arithmetic using the standard basis representation of *S* and *T*. Derive's COL command may help you in entering these vectors.

$\vec{x} = 1\vec{w}_1 + 1\vec{w}_2 + 2\vec{w}_3$ in the *T* basis

or

$\vec{x} = 3\vec{v}_1 + 1\vec{v}_2 - 1\vec{v}_3$ in the *S* basis

Linearly Independent Vectors

You can determine if a set of vectors is linearly independent using the Null_Space command. If a matrix has the given vectors as its columns, then these vectors are linearly independent if the null space of the matrix is the zero matrix (the empty matrix in Derive).